Divergent series: Cesaro, Abel sums and their generalizations

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1 Introduction

Abel famously said in 1828 that "Divergent series are the invention of the devil, and it is shameful to base them on any demonstration whatsoever".I aim to show that this is not the case.We shall extend the notion of series being convergent. Then, under this new definition we shall see that Cesaro and Abel summation satisfies this new definition, with Cesaro sums being means of partial sums, and Abel sums being the limiting value of a power of series of unit radius of convergence when approaching the radius.

In this paper we shall use the notation $s_n = \sum_{k=1}^{n} a_k$ for the partial sums of the sequence (a_n). We say the infinite sum $\sum_{n=1}^{\infty} a_n$ is convergent if the limit of the partial sums exist, i.e $lim_{n\to\infty} s_n$ exists and is finite. In classical analysis a sequence is said to diverge if it does not converge in the usual Cauchy sense.

So under the definition above the sequences

$$1 - 1 + 1 - 1 + 1 + \cdots \tag{1}$$

$$1 + 2 + 3 + 4 + 5 + \cdots \tag{2}$$

are both said to be divergent. This illustrates our need for a more general definition of convergence, as in equation (1) the partial sums seem to oscillate between values 1 and 0, whereas in (2) the partial sums get arbitrarily large. A more general definition of 'sum' will allow us to classify divergent series.

Let V be the vector space containing all real sequences, then following as in [1]. Our aim is to extend the subspace $W \subset V$ which consists of convergent series. Consider the linear operator $\sum : W \to \mathbb{R}$ defined by $\sum(a_n) := \sum_{n=1}^{\infty} a_n$. So our aim is to extend this linear operator to a larger subspace of V containing W [1], which we shall denote as V'.

[1]This isn't a necessary property of the Linear operator, but is desirable. This property makes the summation method a regular method.

We want $\sum$ to be a linear operator in the sense it will satisfy two useful properties of convergence: So if (a_n) and (b_n) are two real sequences in V' and $\sum(a_n) = a$, $\sum(b_n) = b$ then

$$(1) \qquad \sum(a_n + b_n) = a + b$$

and if $\alpha \in \mathbb{R}$ then;

$$(2) \qquad \sum(\alpha a_n) = \alpha a$$

There is a third property that we would want the linear operator to have namely, $\sum(a_n)_{n\geq 2} = a - a_1$, but this property turns out to be very restrictive for some very important operators in [2]. There are examples in this paper,for example Abel sums.

2 Cesaro Summability

We define the Cesaro sum of the sequence (a_n) to be $C_N := \frac{\sum_{i=1}^{N} s_i}{N}$, the mean of the partial sums.

Definition 2.1. Let V_1 be the subspace of V such that $lim_{N\to\infty} C_N$ exists.

Note. The subscript 1 shall become more clear in the next chapter after generalizing the Cesaro method.

Then define a map $T_1 : V_1 \to \mathbb{R}$ by $T_1(a_n) = lim_{N\to\infty} C_N$. It is easy to see that T_1 is a linear map.
Following the example as in [1], if (a_n) is the sequence where $a_{2n-1} = 1$ and $a_{2n} = -1$, which we shall call the oscillating sequence, then we see that the partial sum

$$s_N = \begin{cases} 1 & \text{if } N \text{ is odd} \\ 0 & \text{if } N \text{ is even} \end{cases}$$

So $T_1(a_n) = lim_{N\to\infty} \frac{1}{N} \lceil \frac{N}{2} \rceil = 1/2$.
We want to show that $W \subset V_1$ so that converegent sequences are also
Cesaro summable with the same limit. We follow the proof from [1]. Suppose that $lim_{n\to\infty} s_n = L$. Fix $\epsilon > 0$, so there is a $N \in \mathbb{N}$ such that for all $n > N$, $|s_n - L| < \epsilon$, hence;

$$T_1(a_n) = lim_{k\to\infty} C_{N+k} = lim_{k\to\infty} \frac{\sum_{i=N+1}^{N+k} s_i}{N + k} \tag{3}$$

as $lim_{k\to\infty} \frac{\sum_{i=1}^{N} s_i}{N+k} = 0$. So by the bounds above,

$$L - \epsilon = lim_{k\to\infty} \frac{k(L - \epsilon)}{N + k} \leq T_1(a_n) \leq lim_{k\to\infty} \frac{k(L + \epsilon)}{N + k} = L + \epsilon$$

So as this works for an arbitrary positive ϵ the result is shown. Note that we have shown that W is a strict subset of V_1 from the example of the oscillating sequence.

We are going to show a property of Cesaro sums to show what types of sequences are Cesaro summable.

Lemma 2.1. *If (a_n) is a sequence in V_1 and the sequence of partial sums (s_n) is monotonic, then (s_n) is bounded.*

Proof. Take an increasing sequence (s_n), clearly it is bounded below. Assume the sequence (s_n) is not bounded above, so if $R \in \mathbb{R}$ then $\exists N \in \mathbb{N}$ such that if $n > N$ then $s_n > R$, as s_n is increasing. Now by (3);

$$T_1(a_n) = lim_{k\to\infty} \frac{s_{N+1} + \cdots + s_{N+k}}{N+k} \geq lim_{k\to\infty} \frac{kR}{N+k} = R$$

but as this works for arbitrary R, $T_1(a_n)$ is unbounded so doesn't converge, hence a contradiction. If (s_n) is decreasing, consider $(-s_n)$. □

Corollary. *If $(a_n) \in V_1$ and (s_n) is monotonic, then $(s_n)_{n\geq 1}$ converges.*

Corollary. *If (a_n) is a sequence in V_1 which is strictly positive, or strictly negative then (s_n) is convergent.*

So if we take the contrapositive for the above results we have some tests to show a sequence isn't in V_1. For example on the 1st corollary: If (s_n) is monotonic and isn't convergent, then (a_n) isn't Cesaro summable. Using this test on the sequence $(a_n) = n^{\frac{-1}{2}}$, shows $(a_n) \notin V_1$.

Note. The two corollaries demonstrate the fact that sums that contain strictly positive or strictly negative terms, are convergent in the Cesaro sense if and only if they are convergent in the usual sense. So Cesaro sums extend W into a subspace containing W and sums that contain both positive and negative terms, but are now convergent in the Cesaro sense.

Lemma 2.2. *Suppose that $T_1(a_n) = L$, then for any $i > 0$*

$$T_1(a_i, a_{i+1}, \cdots) = L - \sum_{n=1}^{i-1} a_n$$

Remark. Once proven, this shows that T_1 has the 3rd wanted property of a linear operator.

Proof. A corrected proof from [1] is used. Let s_N be the Nth partial sum of the sequence $(a_n)_{n\geq 1}$, and similarly let t_N be the Nth partial sum of the sequence $(a_n)_{n\geq i}$. Then $t_N = s_{i-1+N} - s_{i-1}$ so;

$$lim_{N\to\infty} \frac{\sum_{n=1}^{N} t_n}{N} = lim_{N\to\infty} \frac{\sum_{n=i}^{i+N-1} s_n}{N} - s_{i-1}$$

Then using the fact that

$$\frac{\sum_{n=i}^{i+N-1} s_n}{N} = \frac{N+i-1}{N} \frac{\sum_{n=1}^{N+i-1} s_n}{N+i-1} - \frac{\sum_{n=1}^{i-1} s_n}{N} \to L$$

Completes the proof. □

Note. If we let $S = 1 - 1 + 1 - 1 + \cdots$, then by the lemma, $T_1(a_2, a_3, \cdots) = S - 1$. By linearity $T_1(a_2, a_3, \cdots) = -S$, so $S = \frac{1}{2}$ which we had before. So we could get a value for the 'sum' of the oscillating sequence just using the 3 wanted properties of the linear operator.

So we have proven some natural properties of a sequence being Cesaro summable, however, we haven't extended W to all of V;

Example 2.1. To show that it has not been extended to all of V, consider the sequence $(a_n) = n$, which has partial sums $s_n = \frac{n(n+1)}{2}$, so

$$T_1(a_n) = lim_{N\to\infty} \frac{\sum_{i=1}^{N} \frac{i(i+1)}{2}}{N}$$

which using the identities $\sum_{i=1}^{N} i^2 = \frac{N(N+1)(2N+1)}{6}$ and $\sum_{i=1}^{N} i = \frac{N(N+1)}{2}$ means

$$T_1(a_n) = lim_{N\to\infty} \frac{(N+1)(N+2)}{6}$$

which doesn't exist. Alternatively we could have just applied (2.1) as the partial sums for the sequence are clearly unbounded.

So, the natural thing to do now is generalize the map T_1 to a map T_k for any integer $k \geq 1$.

3 Holder's Method

Holder's method is an easy generalization of the Cesaro sum.

Definition 3.1. For a sequence (a_n) with partial sum s_n, define the Holder's mean for integer $k \geq 0$ as

$$H_N^k(s_n) = \begin{cases} s_n & \text{if } k = 0 \\ \frac{1}{N} \sum_{i=1}^{N} H_i^{k-1}(s_n) & \text{if } k \geq 1 \end{cases}$$

So we have $T_1(a_n) = lim_{N\to\infty} H_N^1$, from the definition, if this limit exists. We now denote the subset V_k of V to be such that $(a_n) \in V_k$ if $lim_{N\to\infty} H_N^k$ exists and is finite. We now define the map $T_k : V_k \to \mathbb{R}$ by $T_k(a_n) = lim_{N\to\infty} H_N^k$. Note that V_0=W and V_1 is the same as before.
We make an obvious remark from the definitions as said in [2] , which is useful in some simple theorems, regarding Holder summability. Take integer $k, l \geq 0$ then if we denote the sequence $\left(H_N^k(s_n)\right)_{N\geq 1}$ by $H^k(s_n)$ then

$$H_M^k\left(H^l(s_n)\right) = H_M^l\left(H^k(s_n)\right) = H_M^{k+l}(s_n)$$

Now this result immediately proves the following result along with the fact convergent sums are Cesaro convergent, with the same limit.

Lemma 3.1. *(From [2][p.95]). If a sequence $(a_n) \in V_k$ for some positive integer k, then if $l > k$, $(a_n) \in V_l$. More strictly, $T_k(a_n) = T_l(a_n)$.*

Before we give the next theorem we shall give the standard definition of the little-'o' notation.

Definition 3.2. A sequence $a_n = o(b_n)$ if for all $M > 0$ there exists an integer $k > 0$ such that if $n \geq k$ then $-Mb_n < a_n < Mb_n$. We say $a_n = b_n + o(c_n)$ if $a_n - b_n = o(c_n)$.

In reality $o(b_n)$ is more commonly referred to as a set, but we omit this as it's not relevant for our arguments. The definition of $a_n = o(b_n)$ above is equivalent to the definition $(a_n) = o(b_n)$ if $lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and we shall use this definition more commonly.

Theorem 3.2. *If $(a_n) \in V_k$ then $s_n = o(n^k)$ and $a_n = o(n^k)$.*

Proof. The proof is based of that in [2][p.95]. In this proof we write $H_N^k(s_n)$ as H_N^k. As $(a_n) \in V_k$ then $lim_{n\to\infty} H_n^k$ is finite. Now from the definition $H_n^{k-1} = nH_n^k - (n-1)H_{n-1}^k$. Clearly $lim_{n\to\infty} \frac{H_n^{k-1}}{n} = 0$, so $H_n^{k-1} = o(n)$. By induction we can easily show $H_n^{k-i} = o(n^i)$ for $0 \leq i \leq k$, by using the identity $H_n^{k-i} = nH_n^{k-i+1} - (n-1)H_{n-1}^{k-i+1}$ for $n \geq 2$. Hence $s_n = H_n^0 = o(n^k)$ which is half of the result. Finishing the proof we observe $a_n = s_n - s_{n-1}$ for $n > 1$ and $\frac{a_n}{n^k} \to 0$ as n goes to infinity. □

We shall use this theorem to prove that Holders method doesn't extend W to all of V. Consider the sequence $a_n = (-1)^{n+1} n!$ as in [1]. Then $\left|\frac{a_n}{n^k}\right| = \frac{n!}{n^k} \to \infty$ for all $k \geq 0$ so $(a_n) \notin V_k$ for all positive integer k.
However all is not lost as Holders method does extend V_1 to a larger subspace of V. We have already shown that $V_k \subset V_{k'}$ for $k > k'$, but now wish to show it's strict. We consider the case k=1, then using the sequence $(a_n) = (-1)^{n+1} n$ from [1]. So the sequence of partial sums is $(1, -1, 2, -2, 3, -3, \cdots)$

and by the definition of H_N^1

$$H_N^1 = \begin{cases} \frac{N+1}{2N} & \text{if } N \text{ is odd} \\ 0 & \text{if } N \text{ is even} \end{cases}$$

so $(a_n) \notin V_1$ as subsequence of odd terms converge to $\frac{1}{2}$ and even terms to 0.Alternatively we could have argued $lim_{n\to\infty} \frac{a_n}{n}$ doesn't exist. We now want to show $(a_n) \in V_2$ and then consequently in V_k for all $k > 1$. To do this note that

$$\sum_{m=1}^{N} H_i^1 = \sum_{m=1}^{\lceil \frac{N}{2} \rceil} \frac{m+1}{2m}$$

as if N is odd then there is $\frac{N+1}{2}$ non zero terms. If N is even then there are $\frac{N}{2}$ non zero terms. So now

$$H_N^2 = \frac{1}{N} \sum_{m=1}^{\lceil \frac{N}{2} \rceil} \frac{1}{2} + \frac{1}{2m}$$

Clearly

$$lim_{N\to\infty} \frac{1}{2N} \sum_{m=1}^{\lceil \frac{N}{2} \rceil} 1 = \frac{1}{4}$$

We now want to show

$$lim_{N\to\infty} \frac{1}{2N} \sum_{m=1}^{\lceil \frac{N}{2} \rceil} \frac{1}{m} = 0$$

It is clear by use of a diagram that

$$\frac{1}{N} \sum_{m=2}^{N} \frac{1}{m} \leq \frac{1}{N} \int_1^N \frac{1}{x} dx = \frac{ln(N)}{N} \to 0$$

as N goes to infinity, so $(a_n) \in V_2$.
The Holder method satisfies multiple useful properties namely:

Theorem 3.3. *(a) T_k is a linear map for all $k \geq 0$.*
(b) If $(a_n) \in V_k$ then $T_k(a_n)_{n\geq 1} = a_1 + T_k(a_n)_{n\geq 2}$.

Condition (a) is trivial and is true and was one of our wanted properties of a method of summation. Condition (b) is proved in [2][p. 103-104].

4 Abel Summability

In this section we shall define another linear operator which is very important in defining sums of a divergent series. Consider $a : [0,1] \to \mathbb{R}$, defined by

$a(x) = \sum_{n=1}^{\infty} b_n x^n$. We shall call a the Abel sum of (b_n). Let V_a be the subspace of V for which $a(x)$ has a radius of convergence of 1, and for which $lim_{x\uparrow 1} a(x)$ exists, and is some finite $L \in \mathbb{R}$. As in [3], we call L the Abel limit of the sequence (b_n). Then as in [1], define a map $A : V_a \to \mathbb{R}$ defined as $A(b_n) = L$.

By the same derivation as in [3]

$$\sum_{i=1}^{N} b_i x^i = (1-x) \sum_{i=1}^{N} t_i x^i + t_N x^{N+1} \tag{4}$$

where t_n is the partial sum of the sequence (b_n). In the case of the Cesaro sum C_N is the weighted average of the first N terms of the sequence t_n with weight N. Similarly, if in the ideal case where $lim_{N\to\infty} t_N x^{N+1} = 0$ then the Abel sum of the sequence is the weighted average of the t_i 's with weights $(1-x)x^i$.

Lemma 4.1. *If (b_n) is a sequence of real numbers such that $(b_n) \in V_1$ then $(b_n) \in V_a$.*

Remark. We have already shown that $W \subset V_1$ so once the lemma is proven we will have shown that $W, V_1 \subset V_a$.

Proof. Using a slightly modified proof from [3]. From equation (4), we see that;

$$\sum_{i=1}^{N} b_i x^i = (1-x) \sum_{i=2}^{N} [iC_i - (i-1)C_{i-1}] x^i + (1-x) t_1 x + t_N x^{N+1}$$

as $t_i = iC_i - (i-1)C_{i-1}$ for $i \geq 2$

$$= (1-x) \left[\sum_{i=1}^{N-1} iC_i x^i - \sum_{i=1}^{N-1} iC_i x^{i+1} \right] + t_N x^{N+1} + N(1-x) C_N x^N$$

as $C_1 = b_1 = t_1$

$$= (1-x)^2 \sum_{i=1}^{N-1} iC_i x^i + N(1-x) C_N x^N + t_N x^{N+1}$$

Now I expand on [3] by justifying the following details. Here we shall Now if $(b_n) \in V_1$, then $(C_n)_{n\geq 1}$ converges, then as $x < 1$ this means $NC_N x^N$ converges to 0, as $|C_N|$ is bounded above and Nx^N goes to 0 as N goes to infinity by the ratio lemma .Moreover so does $t_N x^{N+1}$, because $t_n = nC_n - (n-1)C_{n-1}$ for $n > 1$. So now,

$$a(x) = (1-x)^2 \sum_{i=1}^{\infty} iC_i x^i$$

Note there are no issues with convergence, as $\sum_{i=1}^{\infty} ix^i$ is convergent and (C_n) is bounded, so the right hand side is convergent by the Weirstraβ M-test. .

Now we know $lim_{n\to\infty} C_n = L \in \mathbb{R}$, and we want to show $A(b_n) = L$, i.e $lim_{x\to 1} a(x) = L$. Fix $\varepsilon > 0$, then let N be a natural number such that if $n \geq N$ then $|C_n - L| < \varepsilon$. We start by using the identity

$$\sum_{i=1}^{\infty} ix^i = \frac{1}{(1-x)^2} \tag{5}$$

for $|x| < 1$, then $L = (1-x)^2 \sum_{i=1}^{\infty} ix^i L$, so

$$a(x) - L = (1-x)^2 \left[\sum_{i=1}^{N-1} ix^i (C_i - L) + \sum_{i=N}^{\infty} ix^i (C_i - L) \right]$$

$\sum_{i=1}^{N-1} ix^i (C_i - L)$ is a finite real number M. For the second term we use $\sum_{i=N}^{\infty} ix^i < \frac{1}{(1-x)^2}$ from (5) as $x > 0$.

$$|a(x) - L| < (1-x)^2 |M| + \varepsilon$$

, so $|A(b_n) - L| < \varepsilon$ for an arbitrary $\varepsilon > 0$ hence $A(b_n) = L$. □

We've already shown the inclusion $W \subset V_1$ is strict, now want to show $V_1 \subset V_a$ is strict. Consider the sequence $1 - 2 + 3 - 4 + 5 - \cdots$ as in [3]. Using the identity (5), $lim_{x \uparrow 1} \sum_{i=1}^{\infty} i(-x)^i = lim_{x \uparrow 1} \frac{1}{(1+x)^2} = \frac{1}{4}$. However, the sequence isn't Cesaro summable as,

$$C_N = \begin{cases} \frac{j}{2j-1} & \text{if } N = 2j-1 \text{ for some} j \geq 1 \\ 0 & \text{if N is even} \end{cases}$$

So then as $lim_{j \to \infty} \frac{j}{2j-1} = \frac{1}{2}$ the limit doesn't exist.

Now we will look at a Tauberian Theorem, more well known as *the* Tauberian theorem. The idea of Tauberian theorems is that if $\sum_{n=1}^{\infty} b_n = b$, then we know that for a general summation method $\sum$, $\sum(b_n) = b$ [2]. However as we've already seen the converse isn't necessarily true. The general format of these theorems is to assume we know something about $\sum(b_n)$ and the behaviour of b_n as $n \to \infty$ then we can deduce something about the convergence of $\sum_{n=1}^{\infty} b_n$.

Theorem 4.2. The Tauberian Theorem. *Take a sequence of real numbers (b_n), and assume $a(x)$ has radius of convergence of 1 and $A(b_n) = L \in \mathbb{R}$. If $b_n = o(\frac{1}{n})$ then $\sum_{n=1}^{\infty} b_n = L$.*

Proof. Proof from [3]. As $a(x) \to L$ as $x \uparrow 1$ then $lim_{N \to \infty} a(x_N) = L$ where $x_N = 1 - \frac{1}{N}$. We want to show that

$$lim_{N \to \infty} \left[\sum_{n=1}^{N} b_n - \sum_{n=1}^{\infty} b_n x^n \right] = 0$$

when $x = 1 - \frac{1}{N}$, as this will prove the result. Fix $\varepsilon > 0$ then as $nb_n \to 0$ there exists an M such that if $n > M$ then $|nb_n| < \varepsilon$.Take $N > M$, we write

$$\left[\sum_{n=1}^{N} b_n - \sum_{n=1}^{\infty} b_n x^n \right] = I_1 + I_2 + I_3$$

where

$$I_1 = \sum_{n=1}^{M} b_n (1 - x^n), \ I_2 = \sum_{n=M+1}^{N} b_n (1 - x^n), \ I_3 = - \sum_{n=N+1}^{\infty} b_n x^n$$

As $N \to \infty$ then $I_1 \to 0$ is obvious. For I_2 we use the well known identity, $(1 - x^k) = (1-x)(1 + x + \cdots + x^{k-1})$ derived from the geometric series. So

$$|I_2| \leq (1-x) \sum_{n=M+1}^{N} |b_n| (1 + x + \cdots + x^{n-1})$$

[2]For any regular method

as $1 - x \geq 0$. Clearly

$$|I_2| \leq (1-x) \sum_{n=M+1}^{N} |nb_n| < (1-x)(N-1-M)\varepsilon$$
$$< N(1-x)\varepsilon = \varepsilon$$

as $1 - x = \frac{1}{N}$. As $|I_2| < \varepsilon$ for arbitrary $\varepsilon > 0$,when N is sufficiently large, this shows $I_2 \to 0$ as N goes to infinity. For the last term

$$|I_3| \leq \sum_{n=N+1}^{\infty} |nb_n| \frac{x^n}{n} < \frac{\varepsilon}{N+1} \sum_{n=N+1}^{\infty} x^n$$

Then $\sum_{n=N+1}^{\infty} x^n < \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} = N$ so

$$|I_3| < \varepsilon$$

hence proving the claim. □

A stronger of Taubers theorem is due to Littlewood, and is proved in [2][p. 154-160]

Theorem 4.3. *For a sequence of real numbers* (b_n)*, if* $a(x)$ *has radius of convergence of 1,* $A(b_n) = L \in \mathbb{R}$*, and if there exists an* $M \in \mathbb{R}$ *such that if* $n|b_n| < M$ *for all* n*, then* $\sum_{n=1}^{\infty} b_n = L$*.*

We now want to compare the Abel and Holder methods. We now define a method based on Cesaro means for summation which is equivalent to Holders method for integer value of k. The reasons for doing this are clear after reading theorems 49 [2][p. 103]and 55, 56 from [2][p. 108]. Cesaro means of a sequence (a_n) are defined as $C_n^k = \frac{A_n^k}{E_n^k}$ where $A_n^0 = s_n$, $A_n^k = \sum_{i=1}^{n} A_i^{k-1}$ and $E_n^k = \begin{pmatrix} n+k \\ k \end{pmatrix}$. It is shown in [2][p. 96] that this definition is equivalent to the definition of the Cesaro mean given in [2]. This seems familiar to the Holders method, however in this case, we do only one division whereas for Holders we do division at each stage.
We omit proofs of theorems 49, 55, 56 from [2].However these results show that Abel's method is strictly stronger than Holder's method for any value of k. So despite being the most natural, Holders is by far not the strongest method.

5 A general method of summation

Following the lines of [1], we now discuss a more general method of summation. So far we have discussed 3 types of summation, and we wish to generate a more general form that will allow us to generalize our methods. We start by denoting the space of analytic functions on the unit interval by A. Analytic functions or smooth functions are functions that are infinitely differentiable on a region. We want this method to allow that convergent series are convergent in this method. Our method is based on integral transforms of the space A.
We denote $f(a, x) = \sum_{n=0}^{\infty} a_n x^n$ for $|x| < 1$, so we have a sequence $(a_n)_{n \geq 0}$ that are the Taylor coefficients for the function f. Consider the class of functions ϕ with the following properties:
(1) There exists an $0 < \alpha_0 < 1$ such that every function $\psi \in \phi$ is analytic in $[\alpha_0, 1)$

(2) If $\psi \in \phi$ then $lim_{x\uparrow 1}\psi(x) = \infty$
(3) For $\psi \in \phi$, $\psi(x)$ is non zero for all $x \in [\alpha_0, 1)$
(4) If we define a sequence $\psi'_{k+1}(x) = \psi_k(x)$ for $k \geq 0$ where $\psi_0 = \psi$, then

$$lim_{x\uparrow 1}\frac{\psi_{k+1}(x)}{\psi_k(x)} = 0$$

So we now define for $f(a) \in A$(The analytic function on A such that $f(a)$ applied to a point x gives $f(a,x)$) and $\psi \in \phi$,

$$M_{\psi_k}(f(a),x) = \begin{cases} lim_{x\to\alpha_0}M_{\psi_k}(f(a),x) & \text{if } x = \alpha_0 \\ \frac{\int_{\alpha_0}^{x} f(a,t)\psi_k(t)dt}{\psi_{k+1}(x)} & \text{otherwise} \end{cases}$$

For $k \geq 0$. It is stated in [4][p.33] that $lim_{x\to\alpha_0}M_{\psi_k}(f(a),x) = f(\alpha_0)$. This is because $f(a,x)$ is continuous at α_0, so $\forall\varepsilon > 0$, take $\delta > 0$ such that $|f(a,x) - f(a,\alpha_0)| < \varepsilon$ for $x \in (\alpha_0 - \delta, \alpha_0 + \delta)$. Then for any x such that $\alpha_0 - \delta < x \neq \alpha_0 < \alpha_0 + \delta$

$$[f(a,\alpha_0) - \varepsilon]\frac{\int_{\alpha_0}^{x}\psi_k(t)dt}{\psi_{k+1}(x)} < M_{\psi_k}(f(a),x) < [f(a,\alpha_0) + \varepsilon]\frac{\int_{\alpha_0}^{x}\psi_k(t)dt}{\psi_{k+1}(x)}$$

then as $\psi'_{k+1} = \psi_k$ the fundamental theorem of calculus the result is shown.

Now if $\exists\psi \in \phi$ such that $lim_{x\uparrow 1}M_{\psi_k}(f(a),x)$ exists and is finite, then the sequence (a_n) is said to be (M,ψ) summable. As we have done many times before we define the subset of V, V_M such that if $(a_n) \in V_M$ then (a_n) is (M,ψ) summable. Then we consider a map T_M from V_m to $\mathbb{R}$ defined by $lim_{x\uparrow 1}M_{\psi_k}(f(a),x)$, when T_M is applied to a sequence (a_n).
If we consider a sequence (a_n) where $lim_{x\uparrow 1}f(a,x) = L$ then it is easy to show $lim_{x\uparrow 1}M_{\psi_k}(f(a),x) = L$, by the FTC as before. Then as $W \subset V_a$ this shows that $W \subset V_M$.
Now we discuss the important subclass of ϕ consisting of functions of the form $\beta_m(x) = (1-x)^{-m}$ for $m > 0$,and $\alpha_0 = 0$. In the case $m = 1$;

$$M_{\beta_1}(f(a),x) = \frac{-\int_0^x \frac{f(t)}{1-t}dt}{log(1-x)}$$

However we are more interested in the case m=2 which we shall soon show to be

$$M_{\beta_2}(f(a),x) = \frac{1-x}{x}\int_0^x \frac{f(a,t)}{(1-t)^2}dt = \sum_{n=0}^{\infty}[\sigma_n(a) - \sigma_{n-1}(a)]\,x^n$$

where $\sigma_n(a) = \frac{\sum_{k=0}^{n} s_k}{n+1}$ (by [4]) so the function $M_{\beta_2}(f(a),x)$ is analytic on $(0,1)$ and we shall denote it as $A_1(f(a),x)$. If a sequence is Abel summable we shall say its $(A,0)$ summable, and if the $lim_{x\uparrow 1}A_1(f(a),x)$ exists then we shall say the sequence is $(A,1)$ summable.

Definition 5.1. We say that a sequence (a_n) is slowly oscillating if $N > M \to \infty$ and $\frac{N}{M} \to 1$ then $a_N - a_M = o(1)$. [4][p. 25-26]

Now we shall give two theorems without proof from [4]. For clarity denote $V_n(a) = \frac{1}{n+1}\sum_{k=0}^{n} ka_k$

Theorem 5.1. Corollary of the generazlized Littlewood Theorem. *Suppose that (a_n) is $(A,1)$ summable, then if there exists a $M \in \mathbb{R}$ such that there is an $N \geq 0$ such that if $n > N$ then $V_n(|a|) < M$ then (a_n) is $(A,0)$ summable.*

Theorem 5.2. Tauberian theorem for the (M, ψ) summation Method. *Take a sequnce* $(a_n) \in V_M$ *and suppose that*

$$lim_{x \uparrow 1} \frac{\psi_1(x) f'(a,x)}{\psi(x)} = 0$$

. *Then if* $(V_n(a))$ *is slowly oscillating then the series* (s_n) *converges to the limit of* $M_\psi(f(a), x)$ *as* x *goes to 1 from below.*

The method of a $(A, 1)$ summability is a generazliation of the normal Abel method $(A, 0)$. This means we can generalize the Abel method by an inductive method as follows. For any $m \geq 1$ define;

$$A^m(f(a), x) := A^1(A^{m-1}(f(a), x))$$

This means the mth power Abel method is applying the $(A, 1)$ method to $A^{m-1}(f(a), x)$. Our aim now is to generate a power series(as in [4]) for $A^1(f(a), x)$ as this will then generalize to the mth power by induction. For a function $f \in A$, consider

$$\frac{f(a,x)}{1-x} = \sum_{k=0}^{\infty} a_k x^k \sum_{k=0}^{\infty} x^k$$

$$= \sum_{k=0}^{\infty} \left[\sum_{j=0}^{k} a_j 1_{k-j} \right] x^k = \sum_{k=0}^{\infty} s_k x^k$$

because of the Cauchy product for power series. By the same argument

$$\frac{f(a,x)}{(1-x)^2} = \sum_{k=0}^{\infty} \left[\sum_{j=0}^{\infty} s_j \right] x^k$$

$$= \sum_{k=0}^{\infty} (k+1) \sigma_k(a) x^k \tag{6}$$

Clearly $\sum_{k=0}^{\infty} (k+1) \sigma_k(a) y^k$ is uniformly convergent on $[o, x]$ by the theorem for Cauchy Product of Power series. This means if we replace x by y we can integrate (6) term by term between 0 and x so

$$\int_0^x \frac{f(a,t)}{(1-t)^2} dt = x \sum_{k=0}^{\infty} \sigma_k(a) x^k$$

This means that

$$A^1(f(a), x) = (1-x) \sum_{k=0}^{\infty} \sigma_k(a) x^k = \sum_{k=0}^{\infty} \sigma_k x^k + \sum_{k=1}^{\infty} \sigma_{k-1}(a) x^k$$

$$= \sum_{k=0}^{\infty} \triangle \sigma_k(a) x^k \tag{7}$$

where for $k \geq 0$, $\triangle \sigma_k(a) = \sigma_k(a) - \sigma_{k-1}(a)$ where we define $\sigma_{-1} = 0$. Now to generalise, let $\sigma_N^m(a) = \frac{\sum_{k=0}^{N} \sigma_k^{m-1}}{N+1}$ for $m \geq 0$ where $\sigma_N^0 = s_n$. Then define $\triangle \sigma_N^m = \sigma_N^m - \sigma_{N-1}^m$ where $\sigma_{-1}^m = 0$.Then if for an inductive arguement, assume that

$$A^{m-1}(f(a), x) = \sum_{k=0}^{\infty} \triangle \sigma_k^{m-1} x^k$$

then by (7)

$$A^m(f(a),x) = A^1(A^{m-1}(f(a),x),x) = \sum_{k=0}^{\infty} \triangle\sigma_k^m x^k$$

If the $lim_{x\uparrow 1} A^m(f(a),x)$ exists and is finite then we say that (a_n) is (A,m) summable, thus generalizing the Abel sum.

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