Arithmetic Ramsey Theory

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Warwick Maths Society

Question

Can one color the natural numbers $\mathbb{N} = \{1, 2, \dots\}$ with three colors so that whenever two numbers have the same color, their sum has a different color?

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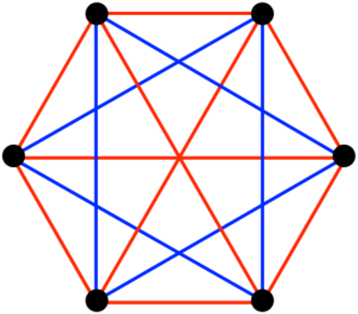
What about other patterns?

Ramsey's theorem

Theorem (Ramsey, 1928)

For every $k, r \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that if the edges of a complete graph on n vertices are coloured with r colors, then there is a monochromatic complete subgraph with k vertices.



$k = 3$, $r = 2$, $n = 6$

Frank Ramsey

Theorem (Schur)

For any finite coloring $\mathbb{N} = C_1 \cup C_2 \cup \cdots \cup C_r$ *there exists* $C \in \{C_1, \ldots, C_r\}$ *and* $x, y \in \mathbb{N}$ *such that* $\{x, y, x+y\} \subset C$.

Issai Schur

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Theorem (Schur, again)

For every r *there exists* $N \in \mathbb{N}$ *such that for every partition* $\{1, \ldots, N\} = C_1 \cup \cdots \cup C_r$ *there exist* $C \in \{C_1, \ldots, C_r\}$ *and* $x, y \in \{1, \ldots, N\}$ *such that* $\{x, y, x+y\} \subset C$.

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Schur used this to show that any large enough finite field contains nontrivial solutions to Fermat's equation $x^n + y^n = z^n$.

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Theorem (van der Waerden)

For every $k \in \mathbb{N}$ and any finite partition $\mathbb{N} = C_1 \cup \cdots \cup C_r$, there exist $C \in \{C_1, \ldots, C_r\}$ and $x, y \in \mathbb{N}$ such that

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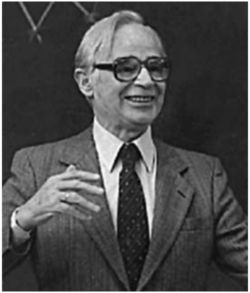

Bartel L. van der Waerden

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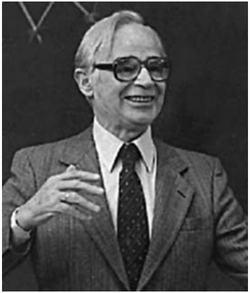

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For every finite partition $\mathbb{N} = C_1 \cup \cdots \cup C_r$ and every finite set $F \subset \mathbb{N}$, there exist $C \in \{C_1, \ldots, C_r\}$ and $x, y \in \mathbb{N}$ such that

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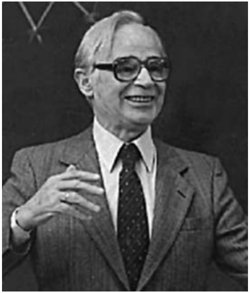

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Theorem

Let G be an infinite abelian group.
For every finite partition $G = C_1 \cup \cdots \cup C_r$ and every finite set $F \subset G$, there exist $C \in \{C_1, \ldots, C_r\}$ and $x \in G$, $y \in \mathbb{N}$ such that

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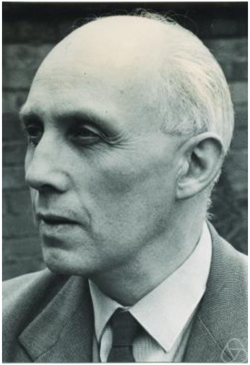

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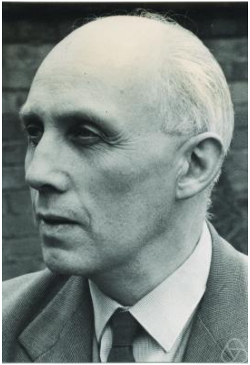

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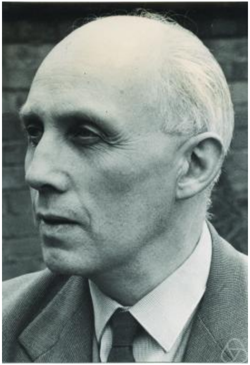

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Observation

Not always the "largest" color possesses the desired configuration. Indeed let

$$C_1 = \{\text{odd numbers}\},\ C_2 = \{\text{multiples of 4}\},\ C_3 = \{x \equiv 2 \bmod 4\}$$

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- Given $E \subset \mathbb{N}$, its upper density is

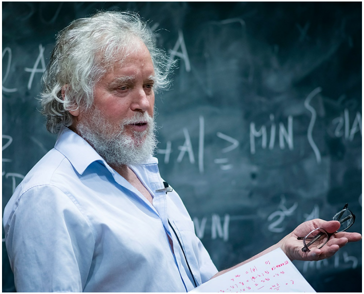
$$\overline{d}(E) := \limsup_{N\to\infty} \frac{\left|E \cap \{1, \dots, N\}\right|}{N}$$

- Upper density is shift invariant: $\bar{d}(E - n) = \bar{d}(E)$ for all n.
- Upper density is subadditive: $\bar{d}(A \cup B) \leq \bar{d}(A) + \bar{d}(B)$.
- In any finite coloring $\mathbb{N} = C_1 \cup \cdots \cup C_r$ some C_i has positive upper density.

Theorem (Szemerédi, 1975)

Let $E \subset \mathbb{N}$ be such that $\overline{d}(E) > 0$. Then for every $k \in \mathbb{N}$ there exist $x, y \in \mathbb{N}$ such that

$$\{x, x + y, x + 2y, \ldots, x + ky\} \subset E.$$



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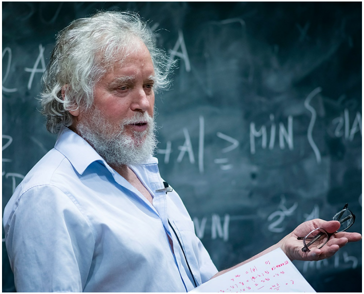

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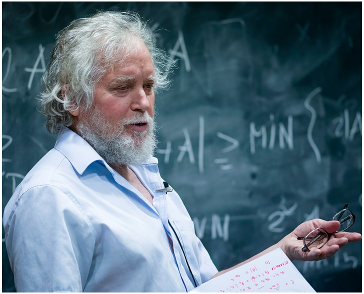

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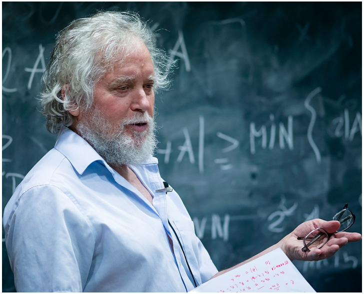

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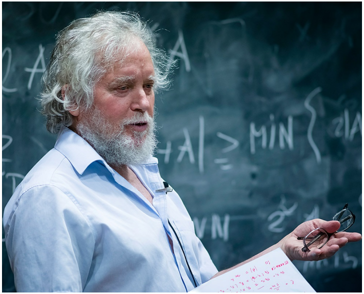

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- In 2004, Green and Tao combine ideas from all these proofs to show that the primes contain arbitrarily long arithmetic progressions.

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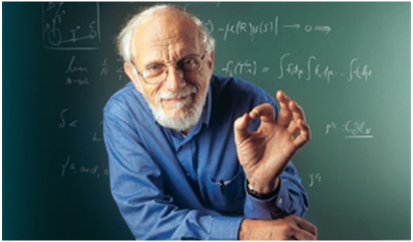
Furstenberg's idea

For $E \subset \mathbb{N}$ and $n \in \mathbb{N}$, let $E - n := \{x \in \mathbb{N} : x + n \in E\}$.

$$\begin{array}{ll} & \exists x, y \in \mathbb{N} \text{ s.t. } \{x, x+y, \ldots, x+ky\} \subset E \\ \iff & \exists y \in \mathbb{N} \text{ s.t. } E \cap (E-y) \cap \cdots \cap (E-ky) \neq \emptyset \\ \Leftarrow & \exists y \in \mathbb{N} \text{ s.t. } \bar{d}\big(E \cap (E-y) \cap \cdots \cap (E-ky)\big) > 0 \end{array}$$



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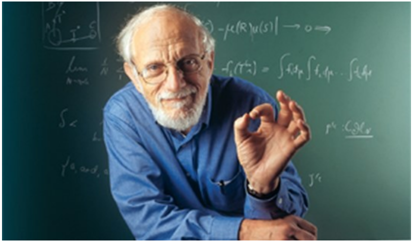

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Corollary

Given $E \subset \mathbb{N}$ with $\bar{d}(E) > 0$ there exists $y \in \mathbb{N}$ such that

$$\bar{d}\big(E \cap (E-y) \cap \cdots \cap (E-ky)\big) > 0.$$

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Polynomial Configurations

Theorem (Furstenberg-Sárközy)

If $A \subset \mathbb{N}$ *has* $\bar{d}(A) > 0$*, then there exist* $x, y \in \mathbb{N}$ *such that* $\{x, x + y^2\} \subset A$.

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Theorem (Bergelson-Leibman)

Let $p_1, \ldots, p_k \in \mathbb{Z}[x]$ *with* $p_i(0) = 0$*. If* $A \subset \mathbb{N}$ *has* $\bar{d}(A) > 0$*, then there exist* $x, y \in \mathbb{N}$ *such that*

$$\{x, x + p_1(y), \ldots, x + p_k(y)\} \subset A.$$

Taking $p_i(y) = iy$ one recovers Szemerédi's theorem.

Conjecture

For every finite coloring $\mathbb{N} = C_1 \cup C_2 \cup \cdots \cup C_r$ *there exist* $C \in \{C_1, \ldots, C_r\}$ *and* $x, y, z \in C$ *such that* $x^2 + y^2 = z^2$*.*

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- We don't even know if there is always a solution with $x, z \in C$ and $y \in \mathbb{N}$!
- W. Sun showed that if the Gaussian integers are finitely coloured, then there is a solution to $x^2 + y^2 = z^2$ with x and z of the same colour.

W. Sun, https://arxiv.org/abs/1405.0241

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- The conjecture was been established when $r = 2$ in 2016, but the proof relies on the use of a computer. The smallest n such that every 2-colouring of $\{1, \ldots, n\}$ yields a monochromatic Pythagorean triple is $n = 7825$, the proof essentially listing all possible colorings occupies 200 terabytes!

W. Sun, https://arxiv.org/abs/1405.0241

M. Heule, O. Kullmann and V. Marek, https://arxiv.org/abs/1605.00723

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- Corollary: $x, y \in \mathbb{N}$ such that $\{x, y, xy\} \subset C$.
  [Proof: restrict the coloring to $\{2^1, 2^2, 2^3, \ldots\}$.]

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Hindman, Trans. Amer Math. Soc., 1979

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- A variation on Hindman's method gives $x = x'$.

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B. Green, T. Sanders, Disc. Anal., 2016

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For every finite coloring $\mathbb{N} = C_1 \cup C_2 \cup \cdots \cup C_r$ *there exist* $C \in \{C_1, \ldots, C_r\}$ *and* $x, y \in \mathbb{N}$ *such that* $\{\quad x+y, xy\} \subset C$.

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J. Moreira, Ann. of Math., 2017

Theorem (M.)

Let $s \in \mathbb{N}$ *and, for each* $i = 1, \ldots, s$*, let* $F_i \subset \mathbb{Z}[x_1, \ldots, x_i]$ *be a finite set of polynomials such that with* 0 *constant term.*
Then for any finite coloring $\mathbb{N} = C_1 \cup \cdots \cup C_r$ *there exist* $C \in \{C_1, \ldots, C_r\}$ *and* $x_0, \ldots, x_s \in \mathbb{N}$ *such that for every* $i, j \in \mathbb{Z}$ *with* $0 \leq j < i \leq s$ *and every* $f \in F_{i-j}$ *we have*

$$x_0 \cdots x_j + f(x_{j+1}, \ldots, x_i) \in C$$

J. Moreira, Ann. of Math., 2017

Corollary

For every finite coloring $\mathbb{N} = C_1 \cup C_2 \cup \cdots \cup C_r$ *there exist* $C \in \{C_1, \ldots, C_r\}$ *and* $x, y, z, t, w \in \mathbb{N}$ *such that*

$$\left\{ \begin{array}{ccccc} x & & & & \\ xy, & x+y & & & \\ xyz, & x+yz, & xy+z & & \\ xyzt, & x+yzt, & xy+zt, & xyz+t & \\ xyztw, & x+yztw, & xy+ztw, & xyz+tw & xyzt+w \end{array} \right\} \subset C$$

Corollary

Let $k \in \mathbb{N}$ *and* $c_1, \ldots, c_k \in \mathbb{Z} \setminus \{0\}$ *be such that* $c_1 + \cdots + c_k = 0$. *Then for any finite coloring of* $\mathbb{N}$ *there exist pairwise distinct* $a_0, \ldots, a_k \in \mathbb{N}$, *all of the same color, such that*

$$c_1 a_1^2 + \cdots + c_k a_k^2 = a_0.$$

In particular, there exist $x, y, z \in C$ such that $x^2 - y^2 = z$.

Infinite configurations

- It follows from Ramsey's theorem that for every finite partition $\mathbb{N} = C_1 \cup \cdots \cup C_r$, there exist $C \in \{C_1, \ldots, C_r\}$ and an infinite set $I \subset \mathbb{N}$ such that

$$I \oplus I := \{i + j : i, j \in I, i \neq j\} \subset C.$$

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- It is not true that every $A \subset \mathbb{N}$ with $\bar{d}(A) > 0$ contains a set of the form $I \oplus I$: take $A = 2\mathbb{N} - 1$.

Conjecture (Erdős)

If $A \subset \mathbb{N}$ has $\bar{d}(A) > 0$, then there exist an infinite set $I \subset \mathbb{N}$ and a shift $t \in \mathbb{N}$ such that $(I \oplus I) + t \subset A$.

Theorem (M.-Richter-Robertson)

Let $A \subset \mathbb{N}$ have $\bar{d}(A) > 0$. Then there exist infinite sets $I, J \subset \mathbb{N}$ such that $I + J \subset A$.

J. Moreira, F. Richter, D. Robertson, Ann. of Math, 2019

Theorem (M.-Richter-Robertson)

Let $A \subset \mathbb{N}$ have $\bar{d}(A) > 0$. Then there exist infinite sets $I, J \subset \mathbb{N}$ such that $I + J \subset A$.

Question

Is it true that every set $A \subset \mathbb{N}$ with positive density contains a set of the form $I + J + K$?

Question

Do the primes contain a sum $I + J$ where $I, J \subset \mathbb{N}$ are infinite?

- Granville showed that yes conditionally on the Hardy-Littlewood tuples conjecture!
- That the primes contain $I + J$ where $|I| = \infty$ and $|J| = 2$ is equivalent to Zhang's theorem that the primes have bounded gaps infinitely often.

J. Moreira, F. Richter, D. Robertson, Ann. of Math, 2019

Further watching/reading

- Numberphile video with Tim Gowers on van der Warden's theorem: `https://youtu.be/kE3OuzlkUnU`
- Numberphile video with James Grime on monochromatic Pythagorean triples: `https://youtu.be/1gBwexpG0IY`
- Website by the authors about monochromatic Pythagorean triples: `https://www.cs.utexas.edu/~marijn/ptn/`
- Clip from QI featuring Ramsey theory: `https://youtu.be/qbInsYok8x8`