Mathematics: The Last Truth of Democracy?

Abstract: *let* C *represent the list of candidates in any given election and* E *be the electorate where* $|E| = n$ *and* $|C| = m$.
Furthermore define the preferential binary operator $c_1 \succ_i c_2$ *for* $c_1, c_2 \in C$ *to represent the* i^{th} *voter preferring candidate* c_1 *to* c_2. *If the overall concesus is candidate* c_1 *over* c_2 *then we denote this* $c_1 \succ_E c_2$. *Let* $T_i: C \rightarrow P_i$ *be the choice function of the* i^{th} *voter where* P_i *represents a ranking of elements* $\{c_1, \ldots, c_m\}$ *and finally let* $F: T(C) \rightarrow P_E$ *be the function of an electoral system where* $T(C) = \{P_1, \ldots, P_n\}$ *and* P_E *represents the output ranking of candidates*

§1. Introduction

Whilst the applications of mathematics to topics like physics and economics are well explored, its applications to politics are often heavily overlooked. After all, the very foundations of democracy in the modern world and built around the idea of numbers; it is not as simple as adding up the number of votes and there are various functions, indices and axioms used in order to construct a working electoral system. Primarily there exists two different ways of classifying electoral systems, the first being preferential vs non-preferential. A preferential system is defined as ranking the candidates from 1 to m; these rankings can be weighted. Conversely a non-preferential system simply gives the voter as many votes as there are for candidates to be elected which is often far less centred around mathematical systems since the only preference relation exists between their choice and those they don't choose. The other way of viewing electoral systems in the eyes of mathematicians such as Konstantinov is that there exists three main groups of electoral systems: "*majoritarian, proportional* and *mixed* [1]*.*" However it is worth noting that mixed is an incredibly broad variety of systems. Bruno Simeone and Friedrich Pukelsheim [2] came up with several criteria to assess both the real world effectiveness and suitability of these systems which namely composed of "transparency and simplicity," "accuracy," forming a parliament that is "capable of governing," ensuring there is not a majority government formed by a "minority of voters," and to "encourage participation." In order to assess these criteria, there needs to be various real world factors to compare them with which can include things like electoral turnout, government success, overall representation and most importantly the possibility of an electoral paradox occurring such that a government comes into power that is either incapable of governing or unrepresentative.

§2. Non-preferential plurality-based electoral systems

To begin, we can consider the electoral systems under the Majoritarian group which is based primarily around the concept of "the candidate who is ranked in first place most often wins." The plurality rule is mathematically defined as if $|\{v_i \in E: x \succ_i y\}| > |\{v_i \in E: y \succ_i x\}|$ then $x \succ_E y$. These systems have their own criteria which can be summarised in around 5-6 rules which a plurality-based electoral system should aim to satisfy.

1. The Majority Criteria which states if one candidate is preferred by a majority of voters, then that candidate must win. This can be summarised as if $|\{v_i \in E: x \succ_i i\ y\}| > ½ * |E|$ implies $x \succ_E y$.
2. The second is the Condorcet criterion which says candidate x should win if for every other candidate Y there is a plurality of voters that answers affirmatively to the question 'Do you prefer X to Y?'. This states that if $x \succ_E \{x'\}$ where $x' = \{x' \in C: x' \neq x\}$ *then* $T_i(C) = \{x, \ldots\}$ for $1 \leq i \leq n$. The inverse of this is the Condorcet loser criterion that will never allow a *Condorcet loser* to win.
    a. A Condorcet loser is a candidate who can be defeated in a head-to-head competition against each other candidate.
3. There is also the Mutual Majority criterion stating if there is a subset S of the candidates, such that more than half of the voters strictly prefer every member of S to every candidate outside of S, this majority voting sincerely, the winner must come from S which mathematically speaking is $if\ S \subset C\ such\ that\ S \xrightarrow{\succ_{E'} (C}$ where $|E'| \geq \frac{1}{2} * |E|\ then\ P \in S$.
4. The Independence of irrelevant alternatives criteria which argues that in order for an electoral system to be democratic, the election outcome remains the same even if a candidate who cannot win decides to run.
5. Finally, there is the Independence of Clones criterion which says the election outcome should remain the same even if an identical candidate who is equally-preferred decides to run.
    a. This is often viewed in context of the Spoiler effect which is defined as when a candidate running on an identical platform affects the chances of a candidate already running.

May's theorem stated that in the event of a two candidate, non-preferential race where there is an odd number of voters then "In a two-candidate election with an odd number of voters, majority rule is the only voting system that is anonymous, neutral, and monotone, and that avoids the possibilities of ties."[1] He also mentioned 3 axioms specific to this case

- Anonymity: each voter is given the same weight i.e. F(E) = F(σ(E)) where σ is a function affecting the magnitude of the elements of E
- Neutrality: relabelling candidates means relabelling the result i.e. if C becomes τ(C) then F(E) becomes τ(F(E)) for τ being some permutation of E
- Monotonicity: if the election is inconclusive and v_i changes preference from c_2 to c_1 then c_1 will win

In order to prove that May's Theorem concludes that the only system satisfying these criteria is a Majority Rule. To prove this, we must first prove the following lemma:

$\boldsymbol{Lemma\ 1.1}$: $if\ a\ voting\ system\ V\ for\ an\ election\ with\ two\ candidates\ is\ anonymous, neutral\ and\ monotone\ then\ it\ is\ a\ quota\ system\ where\ a\ quota\ system.$ [3]

$Here\ a\ quota\ system\ is\ defined\ such\ that\ if\ there\ is\ some\ number\ q\ called\ the\ quota,$

[1] http://pi.math.cornell.edu/~mec/Summer2008/anema/maystheorem.html

$such\ that\ a\ candidate\ will\ be\ declared\ a\ winner\ of\ an\ election\ if\ and\ only\ if\ he\ or\ she\ receives\ at\ least\ q\ votes.$[2] $(Note\ that\ under\ a\ quota\ system, both\ or\ neither\ candidate\ can\ win)$

$It\ is\ sufficient\ to\ prove\ the\ following\ two\ conditions\ and\ the\ result\ will\ follow:$

1. $P\ alone\ is\ the\ preferred\ candidate\ when\ at\ least\ q\ people\ vote\ for\ P$
2. $\frac{n}{2} < q \leq n+1$

$By\ anonymity, the\ system\ is\ invariant\ under\ permutations\ of\ the\ electorate\ hence\ the\ outcome\ is\ dependent\ on\ the\ number\ of\ votes\ for\ P$

$Let\ G\ denote\ the\ set\ of\ all\ numbers\ k\ such\ that\ P\ alone\ is\ the\ preferred\ candidate\ when\ exactly\ k\ people\ voted\ for\ P: if\ G = \emptyset\ then\ P\ alone\ will\ never\ win\ by\ neutrality \Rightarrow q = n+1$

$If\ G\ is\ not\ empty\ then\ let\ q = \inf(G)\ which\ is\ 1\ be\ monotonicity$

$By\ neutrality, if\ k \in G\ then\ n-k \notin G\ else\ the\ opposing\ candidate\ would\ win\ and\ hence\ n-k < k \Rightarrow \frac{n}{2} < k$

$Furthermore\ if\ G\ is\ not\ empty\ then\ q \leq n. Combining\ with\ the\ empty\ case\ it\ follows\ that\ \frac{n}{2} < q \leq n+1$ ■

$\boldsymbol{Lemma\ 1.2}: Lemma\ 1.1\ implies\ May's\ Theorem$ [3]

$The\ system\ V\ satisfies\ the\ 3\ electoral\ axioms\ and\ in\ addition\ is\ a\ quota\ system.$
$In\ order\ to\ prove\ the\ system\ must\ be\ majority\ rule, we\ need\ to\ show\ q = \frac{n+1}{2}$
$Proof\ by\ contradiction; suppose\ q \neq \frac{n+1}{2}$
$Case\ 1: if\ q > \frac{n+1}{2} then\ there\ is\ a\ tie\ when\ there\ are\ \frac{n+1}{2}\ votes\ for\ c_1 and\ \frac{n-1}{2}\ votes\ for\ c_2. In\ this\ case, both\ candidates\ lose\ the\ election$
$Case\ 2: if\ q < \frac{n+1}{2} then\ there\ is\ a\ tie\ with\ the\ same\ number\ of\ votes\ as\ case\ 1\ except\ both\ candidates\ will\ win\ the\ election.$
$Hence\ to\ avoid\ ties, q = \frac{n+1}{2} which\ is\ the\ case\ of\ majority\ rule.$ ■

Whilst May successfully proved that in this specific case, the majority rule was clearly the most suited system, when $|C| > 2$ then the theorem no longer applies. This is because the plurality rule becomes what is known as the pure plurality rule which stated that $|\{v_i \in E: x \succ_i x'\}| > |\{v_i \in E: x' \succ_i x\}|$ (where x' is any candidate opposing x) implies x wins the election. Whilst it seems like not much has changed, there are some major consequences here such as the fact that the minority of voters could now decide the result of the election which encourages tactical voting and discouraged participation, which is clearly in breach of

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the real-world criteria stated in the introduction. Furthermore a candidate could now win on a minimum of $\frac{n}{m} + 1$ votes and since m can be greater than n, it can come down to one vote which creates what we shall later define as a dictator. This led to the establishment of Duverger's Law in 1958 which stated that "the simple-majority single-ballot system favours the two-party system." [4] Whilst he stated a lengthy mathematical proof of this, he stated that other factors were linked to "two forces working together: a mechanical and a psychological factor." [4]

A key example of a non-preferential, plurality based voting system is the first-past-the-post (FPTP) system we use in the UK which involves very little mathematics; though it does use electoral districts and hence to ensure a fair election, the government must "Guarantee redistricting on a regular basis to account for demographic change," which can involve taking into account population dynamics and census data. This in itself led to the largest issue of this system, gerrymandering "the partisan manipulation of electoral district boundaries - has plagued modern democracies since their early times." [2] Clearly this system is more in favour of forming a working government instead of being representative as which can be seen in 1951 where "the British Conservative party was returned to government with a sixteen seat majority in parliament based on 48.0 percent of the popular vote, although Labour won slightly more (48.8 percent) of the vote." [5] FPTP only satisfies the absolute majority criteria and even then, only on a district by district basis, which further goes to show the truth behind Duverger's law. Another good example is the majority run-off system which consists of multiple rounds. In each round at least one candidate is removed until a candidate gets more than 50% or two candidates remain. Two-round majority run-off is specific case in which only top two candidates remain after the first round. This special case was the only plurality electoral system that Duverger claimed "favors multi-partism." [4] Whilst it satisfies majority rule, it still fails to satisfy the Condorcet winner criterion since not all candidates go head-to-head, only the top 2 from the first round. An example of a system which was designed with the Condorcet criteria in mind was Copeland's method which consisted of $\frac{m(m-1)}{2}$ head-to-head match-ups where 1 point was awarded for a win and ½ for tie. This satisfied most of the mathematical criteria except from the independence of irrelevant alternatives since a candidate dropping out could detract points from the winner but not necessarily other candidates.

§3. Preferential plurality-based electoral systems

When looking into preferential, plurality-based voting systems, Duverger's theorem no longer applies and hence we can explore situations in which there are more candidates. However, this does also mean that May's theorem is also void, leading Arrow to propose his own theorem that had his own axioms for a functional electoral system where $C \geqslant 2$ and $F: T(C) \rightarrow P_E$: [6]

1. Unanimity: if c_1 is ranked higher than c_2 for all $v_i \in E$ then $c_1 \succ_E c_2$ in P_E
2. Non-dictatorship: there does not exist a single v_i such that $c_1 \succ_i c_2$ implies that $c_1 \succ_E c_2$
3. Independence of irrelevant alternatives: for two preference profiles $(v_1,\dots,v_n)$ and $(w_1,\dots,w_n)$ if c_1 and c_2 have the same ranking in both then they have the same order in $F(v_1,\dots,v_n)$ as $F(w_1,\dots,w_n)$

As you can see, he regarded the Independence of Irrelevant Alternatives as axiomatic, defining it in a way that could be considered the injectivity of the ranking function T. He went on to prove that it was impossible to satisfy all 3 criteria:

$\boldsymbol{Theorem}$ $\mathbf{1.3}$: $Arrow's\ Impossiblity\ Theorem$ [6]

$The\ theorem\ states\ that\ there\ does\ not\ exist\ a\ voting\ system\ for\ which\ the\ 3\ axioms\ hold. This\ can\ be\ shown\ by\ finding\ an\ abitrary\ system\ V\ that\ satisfies\ all\ 3\ and\ finding\ a\ contradiction.$

$Suppose\ m = 3\ such\ that\ C = \{c_1, c_2, c_3\} and\ that\ c_2 is\ the\ least\ popular\ candidate.$
$By\ unaminity \Rightarrow c_1, c_3 \succ_E c_2. We\ shall\ define\ this\ as\ profile\ 0.$

$Let\ profile\ k\ be\ the\ profile\ for\ which\ the\ first\ k\ voters\ prefer\ c_2\ and\ the\ next\ n - k\ voters\ prefer\ c_1\ and\ c_3.$

$Therefore\ there\ exists\ profile\ K\ where\ c_2 \succ_E c_1\ by\ 1\ vote, which\ will\ be\ the\ 'pivotal\ vote.'$

$By\ IIA, so\ long\ as\ profile\ 0\ has\ c_1 \succ_E c_2\ then\ c_3\ is\ irrelevant\ and\ the\ pivotal\ voter\ is\ k.$

$Let\ E_1 = \{v_i : i \leq k - 1\}\ and\ E_2 = \{v_i : i \geq k + 1\}\ and\ suppose\ that:$
$\quad Every\ v_i \in E_1\ has\ ranking\ c_2 \succ c_3 \succ c_1$
$\quad Every\ v_i \in E_1\ and\ v_k\ has\ ranking\ c_1 \succ c_2 \succ c_3$

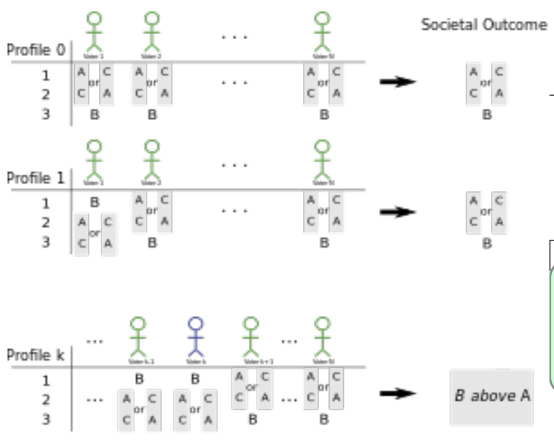
$Hence\ c_1 \succ_E c_2\ since\ it's\ similar\ to\ profile\ k - 1\ except\ by\ unaminity\ c_2 \succ_E c_3$

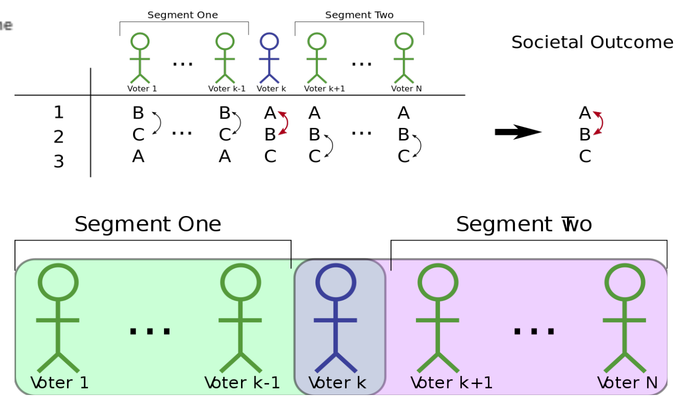
$Now\ suppose\ \ v_k =\ c_2 \succ c_1 \succ c_3\ and\ all\ other\ voters\ move\ c_3 \succ c_2\ but\ c_1\ fixed$
$This\ is\ still\ profile\ k\ with\ c_2 \succ c_1\ and\ by\ IIA\ \ c_1 \succ c_3 \Rightarrow c_2 \succ c_3\ even\ though\ v_k\ was\ the\ only\ voter\ for\ this\ option \Rightarrow v_k\ is\ a\ dictator\ of\ c_2\ over\ c_3.$

$Refering\ back\ to\ the\ original\ profile, the\ pivotal\ voter\ of\ c_2\ over\ c_3\ (v_l) must\ occur\ before\ or\ at\ the\ same\ time\ as\ v_k \Rightarrow l \leq k.$

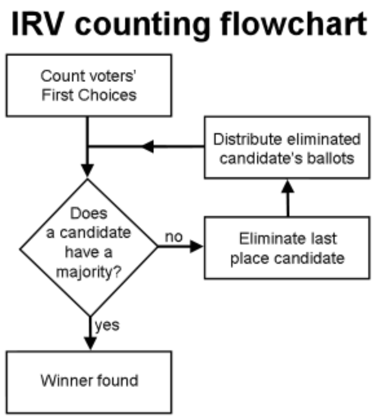
$However\ the\ pivotal\ voter\ of\ c_3\ over\ c_2 (v_{l_2}) must\ also\ come\ before\ the\ dictator\ v_k.$

$Therefore\ l_2 \leq k \leq l. Since\ the\ candidtes\ are\ arbitrary, we\ can\ redo\ profiles\ with\ c_3\ and\ c_2\ switched\ in\ preference\ to\ attain\ l \leq l_2\ which\ means\ they\ are\ the\ same.$
$Therefore, any\ system\ respecting\ unanimity\ and\ IIA\ will\ have\ at\ most\ one\ dictator.$ ■





There are however a few plurality-based preferential systems that still aim to satisfy as many criteria as possible. The main example here would be the Instant Runoff Vote system which is similar to a multi-round runoff except all preferences are taken in to account and the votes are distributed accordingly as demonstrated by the flowchart. In practical terms, this saves on having to run multiple elections. Whilst this system satisfies the Condorcet loser, Clones, majority and mutual majority criterions, it fails Condorcet or irrelevant alternatives for similar reasons to all variations of a run-off voting system.



§4. Non-preferential proportional electoral systems

Now to look at the converse which are systems of proportionality; we begin yet again analysing the non-preferential systems. Proportional representation is defined as a system in which divisions in an electorate are reflected proportionately in the elected body. [7] Define S to be the number of seats available in a body or position and let the numbers of votes and seats for party C_k be $v_k \in \mathbb{N}$ and $s_k \in \mathbb{N}_0$ respectively. Since we are examining a different set of electoral systems, there is a different basis for proportional systems. there must exist a minimum barrier B for a party to qualify for a seat else anyone running would get representation which would cause a failure since S is a subset of the natural numbers. [8] For example, in France the barrier is 12.5%. As a result of this, B≤V/n where V is the total votes and n is the total parties otherwise it is possible for no party to meet the barrier and hence no candidate would be elected. The *proportional system*, or the rule that transforms votes into seats, is described by the vector function $(v_1,\dots,v_n) \rightarrow (s_1,\dots,s_n)$ denoted as $f = (f_1,\dots,f_n)\colon \mathbb{N}^n \rightarrow \mathbb{N}_0^n$. [8]Since S is the total number of seats it follows that $\|f(v)\| = \sum_{k=1}^{n} f_k(v)$= S. Unlike a plurality system, proportionality is based on the idea of more votes equalling more seats and hence $v_j > v_k$ implies that $s_j > s_k$. and furthermore, by proportionality $\frac{s_k}{v_k} \simeq \frac{S}{V}$. The deviation from proportionality is a function $d_f(v) = \left\| \frac{f(v)}{S} - \frac{v}{V} \right\|$ which the system aims to minimise in order to be as representative as possible. Again, since we are looking at a different family of electoral systems, they are also subject to their own properties or rather paradoxes which the system should seek to avoid: [9]

1. Firstly, there is the Alabama Paradox which states that should the number of seats in a body, it does not decrease the number of seats currently held by a party which can be expressed as $f(v,S) \leq f(v,S+T)$ for some $T \in \mathbb{N}$.
2. Next there is the New Party Paradox which argues if there be a new party C_{n+1} such that $\hat{s} = (\widehat{s_1}, \dots, \hat{s}_{n+1})$ where $\hat{s}_{n+1} \geq 1$ with $v = (v_1,..,v_n,v_{n+1})$ then any old party should not gain seats; that is $\widehat{s_k} \leq s_k \ \forall k \in [1,n]$.
3. Thirdly, the Population paradox states if the votes of C_1 and C_2 increase by $u_1 \geq u_2 \geq 1$ then $\hat{s}_1 \geq s_1$ and $\hat{s}_2 \leq s_2$.
4. The Coalition Paradox declares that if C_{n-1} and C_n merge to form a coalition then no other party shall change seats which mathematically means $\widehat{s_k} = s_k \ \forall k \in [1, n-2]$.
5. Finally, the Hare Quota critically states that if we let $\sigma_k = S * \frac{v_k}{V} = \frac{v_k}{q_H}$ where q_H is the Hare quota, then $\lfloor \sigma_k \rfloor \leq s_k \leq \lfloor \sigma_k \rfloor + 1 \ \forall k \in [1,n]$.

Denominator	1	2	3	4	5	6	7	8	Seats won (*)	Proportionate seats
Party A	**100,000***	**50,000***	**33,333***	**25,000***	20,000	16,666	14,286	12,500	**4**	3.5
Party B	**80,000***	**40,000***	**26,666***	20,000	16,000	13,333	11,428	10,000	**3**	2.8
Party C	**30,000***	15,000	10,000	7,500	6,000	5,000	4,286	3,750	**1**	1.0
Party D	20,000	10,000	6,666	5,000	4,000	3,333	2,857	2,500	**0**	0.7
Total									**8**	**8**

However once again it was proven that such a system that satisfied all these qualities does not exist. Balinski and Young published an Impossibility Theorem [6] in 1980 showing that many of these rules were mutually exclusive to each other. Therefore, different electoral systems will aim to satisfy different criteria. Hence the different of proportional non-preferential systems will all differ slightly. For example, D' Hondt's Highest Averages Method, which is used in the United States House of Representatives, uses an algorithm that consists of creating a ratio of votes to seats: [11]

- Order the *m* parties by the rule $v_1 \geq v_2 \geq \cdots \geq v_m$ (if there are parties with equal votes the ordering is by ties).
- Construct the *S* x *m* matrix with elements $h_{i,j} = \frac{v_j}{i}$ $i = \{1,2,\ldots,S\}$ $j = \{1,2,\ldots,n\}$
- Mark the first *S* greatest elements $h_{i,j}$ in descending order of magnitude starting with $h_{1,1}$. If there are equal elements $h_{i,j}$ = $h_{k,jl}$ for |*i*-*k*|+|*j*-*l*| ≥ 1 then first comes the element $h_{i,j}$ when *i*<*k*, or when *i*=*k* and *j*<*l*.
- Determine s_k, *k* = 1,2,...,*n*, as the number of marked elements $h_{i,j}$ with *k* = *j*. All *S* seats are allocated.
- The table below illustrates an example of dividing 8 seats by 4 people, representing the matrix as an 4x8 table of values

§5. Preferential proportional electoral systems

Preferential proportional electoral systems follow the same criteria as non-preferential rules as before against paradoxes however tend to be even more representative as it accounts for the actual ranking of candidates. The most well-known example would be the Single Transferable Vote which is essentially a multiple candidate scenario of IRV. if a candidate receives $v_k \geq \left[\frac{V}{S+1} + 1\right] = q$ votes where S is the number of candidates being elected then they are automatically elected. If $v_k < q$ then the surplus of candidate C [V(C)] is defined as $v_k - q$ and these votes are then transferred to active candidates. It does indeed work incredibly similarly to the IRV however it can be applied to the elections of multiple candidates.

§6. Mixed electoral systems

Finally, a mixed system which combines elements of a plurality/majoritarian system with aspects of a Proportional electoral system in such a way that a voter can have an influence on both parts of the system. [5] These mostly consist of examples using Parallel Voting which is where two systems are used together but some seats are done by the plurality system and some by the proportional system. Take for example the Supplementary Member system in which some seats are assigned through FPTP and the remainder through closed list proportional representation or the Mixed-Member Proportional system where people get two votes, one for their rep and one for their party. These systems aim to take positive aspects of both systems to help provide the best compromise between the two distinct kinds of electoral systems.

§7. Final Remarks

Between the theorems of Duverger, Arrow and Balinski and Young it may appear that maths has only proven to us that there is no perfect electoral system and that is correct to some extent. However, like most subjects where mathematical cases are applied to the real world we must view their usage on a case by case basis. A country such as the United States has what can be considered a two party system and hence a plurality-based non-preferential method would work well whereas in Germany there are a diverse range of political parties aiming to represent subtler distinctions between the views of their constituents which clearly has leanings towards a proportional preferential method. However we must remember that whilst we can try to improve the system, we cannot always improve the candidates. As Robert Byrne put it “Democracy is being allowed to vote for the candidate you dislike least.”

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