Chromatic Polynomials

MA213 - Second Year Essay

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1 Introduction

If one were to delve into the fascinating combinatorial branch of mathematics that is graph theory, it would not take too long at all for one to encounter the infamous four colour theorem. It was proposed that for any planar graph, it would always be possible to colour any vertex in one of four colours such that no pair of vertices connected by a shared edge would be of the same colour. This theorem, formally proposed in 1852, had been left unproven for years, baffling the minds of great mathematicians for over a century, until it was rigorously proven in 1976 by Kenneth Appel and Wolfgang Haken, with the aid of computer calculation and algorithms which were obviously not to hand at the time [2].

In the decades that the proof was left undiscovered, many mathematicians have tried to tackle the problem with various approaches. One attempt proposed by George David Birkhoff in 1912, was the introduction of chromatic polynomials, a set of polynomials in variable λ which could be used to define the number of possible colourings of a given graph using λ colours. If Birkhoff could prove that every chromatic polynomial of a planar graph was strictly greater than zero in the case where $\lambda = 4$, then each planar graph would have at least one 4-colouring, which would prove the four colour theorem to be correct [1].

Sadly, Birkhoffs efforts were unfruitful for their initial goal. However, chromatic polynomials have been since further studied, with mathematicians achieving results that will be shown in this very paper, which have been helpful in finding the number of colourings for various graphs, compounds and augmentations.

2 Fundamentals of Chromatic Polynomials

As given in the introduction, chromatic polynomials focus on the number of ways to "colour" graphs, i.e. how many maps can be made from the vertices of a graph to a number of colours such that no pair of adjacent vertices (vertices that share an edge) are mapped to the same colour. Firstly, it would be ideal to define some concepts and notations that will be commonly used throughout this essay.

2.1 Common Notation

V(G)	The set of vertices of a graph G.
E(G)	The set of edges of a graph G.
P(G,λ)	The number of colourings of a graph G in λ colours.

2.2 Foundations

Definition 1. A *colouring* of a graph G is a map ϕ: V(G) $\rightarrow$ {1, 2, ... λ} such that $\forall$ u,v $\in$ V(G), uv $\in$ E(G) $\Rightarrow$ ϕ(u) $\neq$ ϕ(v) [3,ppXXII-XXIII].

Definition 2. A *colour class* of a colouring ϕ on graph G is a set of vertices $V_i \subseteq$ V(G) such that $\forall$ v $\in V_i$, ϕ(v) = i for a specific i $\in$ {1, 2, ... λ}, i.e. a subset of vertices which are all coloured the same [3,pXXIII].

It is quite clear to notice that the colour classes of a map partition a graph's vertices, as all v $\in$ V(G) are mapped by ϕ, but no vertex can be mapped to two different colours. Thus, by the definition of a partition, $V_1 \cup ... \cup V_\lambda$ = V(G) and $V_i \cap V_j = \varnothing$ $\forall$ i,j $\in$ {1, 2, ... λ}.

Remark 1. The number of colour classes, p, obtained from ϕ is always such that p $\leq \lambda$. Each colour class of a colouring corresponds to a different colour, i.e. $\exists\psi$: {1, 2, ... p} $\rightarrow$ {1, 2, ... λ}, an injective map. As the injection is between two finite sets, the set of colour classes must not exceed the set of colours for injectivity to hold. Thus, p $\leq \lambda\forall\phi$.

By understanding the notion of colouring graphs and the idea of partitioning vertices into those which share the same colour, we can begin to understand how the number of ways a graph can be coloured is representable as a polynomial, as detailed in the following theorem:

Theorem 1. *The number of colourings P(G, λ) is a polynomial in variable λ [4,p7].*

Proof. Let ϕ be a colouring that partitions V(G) into p $\leq \lambda$ colour classes. The 1st colour class, V_1, can be assigned one of λ colours. V_2 cannot be assigned the same colour as V_1, so can be assigned one of λ - 1 colours. Repeating the process $\forall$ V_i, the total number of ways to colour the colour classes = $\lambda(\lambda-1)$ ... $(\lambda-p+1) = \frac{\lambda!}{(\lambda-p)!}$, which is clearly a polynomial in λ. Since V(G) is a finite set, there can only be finite possible partitions of V(G) to satisfy a colouring of the graph G. So P(G, λ) = $\sum_{i=1}^{N} \lambda_i$, where λ_i is the number of ways to assign colour classes to a partition created by ϕ_i, N $< \infty$. Since the number of colourings in any partition can be given as a polynomial in λ, the total number of colourings is a finite sum of polynomials, and is therefore a polynomial itself [4,p7]. ∎

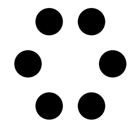
As we can see, the number of colourings in any graph can be represented by a polynomial, formally known as the *chromatic polynomial* of the given graph, which is what this essay will be focusing on. The given theorem also leads nicely to another standard result on these polynomials:

Corollary 1. *P(G, λ) has a degree equal to the order of G [4,p7].*

Proof. Given that we can partition the vertices into p colour classes, in the case where p = λ, then the number of ways those colour classes can be assigned is $\lambda(\lambda-1)$ ... $(\lambda - (\ \lambda - 1)\ )$ ways, which itself is a polynomial of degree λ. The maximum value we can choose for λ is the number of vertices in G, being the order of G, so with n colours (where n is the order of G) and a partition which has n colour classes, there must exist a polynomial in the finite sum of polynomials that forms P(G, λ) that has a degree of n = order of G, being the polynomial given earlier of n products. Since one cannot partition a graph of n vertices into more than n parts, there cannot exist a polynomial in the sum with a greater degree as we cannot assign more than n colour classes. Therefore, the degree of P(G, λ) = order of G, for the finite sum that forms the polynomial contains a polynomial of degree n and no polynomial of a higher degree [4,p7]. ∎

So given a finite graph of any order, one can easily identify how big the polynomial is going to be, degree-wise. However, calculating such a polynomial for any graph may be a tedious process given relatively complex and unusual graphs, for the algorithm used to calculate the general chromatic polynomial can be quite a tedious process, as seen with processes detailed later in this essay. Nevertheless, there are some standard examples of graphs for which there exists a set result for their chromatic polynomial.

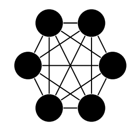
3 Examples of Chromatic Polynomials for Specific Graphs



Example 1. The chromatic polynomial of O_n, the empty graph of order n, is the following [5,p55]:

$$P(O_n, \lambda) = \lambda^n$$

Proof. O_n has n vertices to be coloured. Since $E(O_n) = \varnothing$, no vertex shares an edge with another vertex so a colouring on O_n has no restrictions. Therefore, any vertex can be coloured in one of λ ways regardless of the colour of any other vertex. With λ ways to colour each of the n independently coloured vertices, the total number of colourings for O_n is quite clearly λ^n, the product of the possible colourings of all vertices [5,p55]. ■

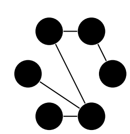


Example 2. The chromatic polynomial of K_n, the complete graph of order n, is the following [5,p54]:

$$P(K_n, \lambda) = \frac{\lambda!}{(\lambda - n)!}$$

Proof. Take $v_1 \in V(K_n)$. This vertex can quite clearly be coloured in λ ways. Since all vertices in K_n share an edge with any other vertex in K_n, then regardless of the choice for $v_2 \in V(K_n)$, it will share an edge with v_1, so it cannot be coloured the same. This means there are λ - 1 ways to colour v_2 given a fixed colouring of v_1. Similarly, choosing $v_k \in V(K_n)$ for some k $\in \{3, 4, ..., n\}$, will always be a vertex sharing an edge with all vertices v_1, ..., v_{k-1}. As any two vertices from v_1, ..., v_k share an edge with each other, v_k can only be coloured in $\lambda - (k-1)$ ways $\forall$ k (again, after fixing colours for v_1, ..., v_{k-1}). Thus, the product of the number of possible colourings for all vertices in K_n is $\lambda(\lambda - 1)$ ... $(\lambda - n + 1) = \frac{\lambda!}{(\lambda-n)!}$ [5,p54]. ■

It follows from this result that the chromatic polynomial of K_n has roots consisting exclusively of the set 1, 2, ... n - 1. Therefore, K_n is only colourable with at least n colours. Knowing that every planar graph can be coloured in 4 colours or less, we can establish that every complete graph of at least 5 vertices cannot be planar. Mind you, it would probably be easier to prove this result alternatively than to try and prove the four colour theorem, as Birkhoff would probably tell you if he were alive today.



Example 3. The chromatic polynomial of T_n, which denotes any tree of order n, is the following [1]:

$$P(T_n, \lambda) = \lambda(\lambda - 1)^{n-1}$$

Proof. Take $v_1 \in V(T_n)$. This vertex can quite clearly be coloured in λ ways. Choose $v_2 \in V(T_n)$ such that $v_1v_2 \in E(T_n)$. Such a choice is possible since T_n is a tree, which implies it is a connected graph, so $\forall$ v $\in V(T_n)$, $\exists$ e $\in E(T_n)$ incident on v. Since v_1v_2 exists, the vertices cannot be coloured the same. This means there are λ - 1 ways to colour v_2, given a fixed colouring of v_1. Similarly, choose $v_k \in V(T_n)$ for some k $\in \{3, 4, ..., n\}$, such that v_k shares an edge with at least one vertex from the set $\{v_1, ..., v_{k-1}\} \subset V(T_n)$. v_k can only be adjacent to one vertex in the set as otherwise, a cycle would be created consisting of v_k, two vertices in $\{v_1, ..., v_{k-1}\}$ that are adjacent to v_k and the path between those two vertices (such a path exists because the subgraph of T_n created by the vertex set $\{v_1, ..., v_{k-1}\}$ and any edges incident with any two vertices in the set is a connected subgraph by choice of vertices). Since v_k is only adjacent to one vertex in the set, it can be coloured in λ - 1 ways $\forall$ k, again given fixed colourings for $v_1, ..., v_{k-1}$. Thus, the product of the number of possible colourings for all vertices in T_n is $\lambda(\lambda - 1)^{n-1}$. ■

4 Augmentations of Graphs

4.1 Initial Ideas

It is possible for graphs to be augmented in order to consider its chromatic polynomial, by adding edges or contracting pairs of vertices to obtain more recognisable graphs. The augmentations we will be using are defined as such:

Definition 3. We denote the graph G + xy as the new graph formed from G by adding a new edge, xy $\notin$ E(G), incident to vertices x,y $\in$ V(G) [3,pXX].

Definition 4. We denote the graph G - xy as the new graph formed from G by removing the already-existing edge, xy $\in$ E(G), incident to vertices x,y $\in$ V(G) [3,pXX].

Definition 5. We denote the graph G $*$ xy as the new graph formed from G by contracting vertices x,y $\in$ V(G) i.e. by removing y from V(G), changing any edge in the form zy $\in$ E(G) to be zx instead $\forall$ z $\in$ V(G), and removing any duplicate edges or loops in E(G) so that the graph is simple [3,pXX].

On a sidenote, a duplicate edge of a graph is an edge which is incident on two vertices that are already adjacent by some other edge, and a loop is an edge that is incident twice on the same vertex. Simple graphs do not contain either of these, as they are exclusively features of multigraphs, and this essay only deals with simple graphs throughout, so precautions have to be taken to ensure that any graph we consider is simple.

Despite going a bit off-topic here, it's interesting to note that duplicate edges pose no further restrictions on colourings since the two vertices they are incident on are already adjacent, so multigraphs containing duplicate edges are can be coloured the same with their duplicate edges removed. However, as loops are edges incident twice on the same vertex, and any given vertex cannot be mapped to two distinct colours, then vertices with loops can never be coloured, thus there does not exist a colouring of any graph containing a loop.

4.2 Relations Between Augmented Graphs

When counting the number of colourings of a graph G, we can consider two vertices, x,y $\in$ V(G), that are not adjacent in G. Any colouring of G can fall into two categories, colourings which do not map x and y to the same colour and colourings which do.

If x and y are mapped to different colours, then we can treat the graph G as G + xy. This is because G + xy follows exactly the same conditions to be coloured as G does, except that x and y can no longer be the same colour, as they are now adjacent vertices. This means that P(G + xy, λ) represents the number of colourings of G such that x and y are different colours.

Alternatively, if x and y are instead mapped to the same colour, then we can treat the graph G as G $*$ xy. As x and y are coloured the same, they can be treated as the same vertex for the sake of colourings and no vertex adjacent to either x or y can take the same colour. Therefore, G $*$ xy follows the same conditions to be coloured as G does if x and y are coloured the same. This means that P(G $*$ xy, λ) is the number of colourings of G such that x and y are the same colour.

These two considerations build up the logic for this very handy theorem:

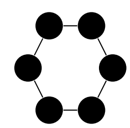
Theorem 2. *P(G, λ) = P(G + xy, λ) + P(G $*$ xy, λ) [5,pp55-56]*

The proof for the above theorem is obvious from considerations of non-adjacent vertices x and y. This can potentially come in handy for finding the chromatic polynomials of more unusual graphs. It is possible to calculate the chromatic polynomial of a graph which isn't one of the basic classifications given earlier, by adding edges and contracting vertices to turn it into more basic graphs. The above theorem also evidently leads to the following result:

Corollary 2. *P(G, λ) = P(G - xy, λ) - P(G $*$ xy, λ) [5,p57]*

Proof. If we let H = G + xy, then evidently G = H - xy. As the only difference between G and H is the existence of one edge, then contracting the vertices together so no edge can exist between them will augment both graphs to give the same result. Thus, G $*$ xy = H $*$ xy. Substituting in the new graphs and rearranging the addition equation gives the subtraction equation. ■

This corollary is actually integral in finding the general case for the chromatic polynomial of cycles inductively, as demonstrated below:



Example 4. The chromatic polynomial of C_n, which denotes the cycle of order n, is the following [1]:

$$P(C_n, \lambda) = (\lambda - 1)^n + (-1)^n(\lambda - 1)$$

Proof. This proof is best given inductively. Cycles require at least 3 vertices, so the first case to be considered is n = 3. Since $C_3 = K_3$, then we have:

$$\begin{aligned} P(C_3, \lambda) &= P(K_3, \lambda) \\ &= \frac{\lambda!}{(\lambda - 3)!} \\ &= \lambda(\lambda - 1)(\lambda - 2) \\ &= (\lambda - 1)^3 - (\lambda - 1) \end{aligned}$$

Thus, the result for the case n = 3 is true. If we assume that the result is true for n = k - 1, we can use the previous corollary to show P(C_k, λ) = P(C_k - xy, λ) - P(C_k $*$ xy, λ). We can make a choice of x,y $\in$ V(C_k) such that x and y are adjacent, which would mean that C_k - xy = T_k, as removing one edge creates a path, which is also a tree. It also means C_k $*$ xy = C_{k-1}, as contracting two adjacent vertices would just shorten the cycle by 1 one vertex. This means that by the corollary, we have:

$$\begin{aligned} P(C_k,\lambda) &= P(T_k,\lambda) - P(C_{k-1},\lambda) \\ &= \lambda(\lambda-1)^{k-1} - (\lambda-1)^{k-1} - (-1)^{k-1}(\lambda-1) \\ &= (\lambda-1)^{k-1}(\lambda-1) + (-1)^{k}(\lambda-1) \\ &= (\lambda-1)^{k} + (-1)^{k}(\lambda-1) \end{aligned}$$

Hence, if the result is true for n = k - 1, it is also true for n = k. Since it is true for n = 3, it is true for all n $\geq$ 3 by induction. ∎

5 Chromatic Reduction

5.1 Chromatic Polynomial Forms - Power Form

We have already proven that the number of colourings of any given graph can be expressed as a polynomial in λ. However, there is another way to imagine this idea using the relation given in the corollary of the last section:

$$P(G,\lambda) = P(G - xy,\lambda) - P(G * xy,\lambda)$$

Taking any graph G, we have already seen the process of splitting its polynomial into the difference of the polynomials of the two augmented graphs given in the result shown above. We can then split the two resultant polynomials of the augmented graphs into even more polynomials using the same relation, as long as the graph has an edge to remove, repeating the process as many times necessary. The polynomial of a graph G can only no longer be split into two polynomials by this relation if it has no edges, as we can no longer define G - xy. However, a graph with no edges is defined as O_n, with n being its order. This means that if we continue to split the chromatic polynomial of any graph G with order n as many times as possible, the polynomial can be expressed as a sum of the polynomials of finitely many empty graphs of varying orders [5,pp57-58]:

$$P(G,\lambda) = \sum_{i=1}^{n} a_i P(O_i,\lambda) = \sum_{i=1}^{n} a_i \lambda^i$$

Where $a_i \in \mathbb{R}$. The form of a chromatic polynomial given above is known as its *power form* as the polynomial is expanded completely into powers of λ. It is easy to see that the coefficients for each power of λ correspond to the number of graphs derived from G from this process of the same order as the power of λ. Note that it isn't equal to the number of graphs derived of the given order, since a negative sign is introduced every time two vertices are contracted.

From this form, we can notice some clear facts about chromatic polynomials:

Theorem 3. *The constant coefficient of a chromatic polynomial, a_0, is always equal to 0 [5,p63].*

Proof. This is obvious as every chromatic polynomial is shown to be the sum of polynomials of empty graphs. Since all empty graphs have at least one vertex, there is no empty graph whose chromatic polynomial is a constant, so there is no constant coefficient in the power form. ∎

Theorem 4. *The leading coefficient of a chromatic polynomial, a_n is always equal to 1 [5,p63].*

Proof. When we split the polynomial of a graph G into G - xy and G * xy, we can see G - xy retains its order, but G * xy has its order one less than G since two vertices have been contracted. This means that the only augmented graph derived from G that has the same order as G is the one created only by removing edges (with no contractions) until it becomes O_n, where n is the order of G. Any other graph derived from augmenting G will have had 2 vertices contracted at some point, causing its degree to be less than n. When we split the polynomial, we also see that the polynomial for G - xy remains positive, so since only one graph augmented from G retains G's order, in which its polynomial remains positive under reduction, the coefficient a_n can only be 1. We know this is the leading coefficient since the degree of a chromatic polynomial is equal to its graph's order. ■

We denote the process of splitting a chromatic polynomial into the sum of the polynomials of either empty or complete graphs as *chromatic reduction*. This process is quite possibly the nicest algorithm to calculate the chromatic polynomial for any graph G, though the more vertices a non-trivial graph has, the more the algorithm becomes fiddly and tedious, especially for graphs with a lot of edges that have to be removed or contracted. The reduction involving empty graphs has been demonstrated above, whereas complete graph reduction is details below, which would be more ideal for graphs that contain more pairs of adjacent vertices than non-adjacent pairs of vertices.

5.2 Chromatic Polynomial Forms - Factorial Form

Similarly with power form, we can reintroduce an earlier relation and consider repeating it over and over again. We saw this relation earlier:

$$P(G, \lambda) = P(G + xy, \lambda) + P(G * xy, \lambda).$$

As with calculating power form, we can repeatedly split the polynomials of the new graphs obtained into more augmented polynomials by adding edges and contracting vertices, in a different variant of chromatic reduction. This time around, the polynomial of a graph G can no longer be split using this method if an edge already exists between any pair of vertices in G, as we can no longer define G + xy. However, a graph with every vertex adjacent to all other vertices is defined as K_n, with n being its order. This means that if we continue to split the chromatic polynomial of any graph G with order n as many times as possible, the polynomial can be expressed as a sum of the polynomials of finitely many complete graphs of varying orders [5,pp56-57]:

$$P(G, \lambda) = \sum_{i=1}^{n} b_i P(K_i, \lambda) = \sum_{i=1}^{n} b_i \lambda_{\{i\}}$$

Where we define $\lambda_{\{i\}} = \frac{\lambda!}{(\lambda - i)!}, b_i \in \mathbb{R}$. This form of the polynomial is known as *factorial form*, as the form is the sum of fully factorised polynomials. Since every term is a multiple of λ, the sum agrees that the constant coefficient of all chromatic polynomials is equal to 0. There can also be only one augmented graph created from the process to retain the same order as the original graph as well, which concurs with the theorem that the leading coefficient is equal to 1 under the same ideas.

The factorial form will come in handy in later proofs, but currently with normal binary operations, the $\lambda_{\{i\}}$ term seems difficult to manipulate and retain a nice value. We shall define a new binary operation in order to make factorial form easier to manage:

Definition 6. Let $\lambda_{\{i\}} = \frac{\lambda!}{(\lambda-i)!}$. Let A and B be sums of such factorial polynomials with real coefficients, A = $\sum_{i=1}^{m} a_i\lambda_{\{i\}}$ and B = $\sum_{j=1}^{n} b_j\lambda_{\{j\}}$. We define the *umbral product* of A and B, denoted as A $\otimes$ B, as the multiplication of the polynomials A and B but instead, the factorials of λ are multiplied by summing the indices denoted to each λ rather than the powers of λ (i.e. $\lambda_{\{i\}} \otimes \lambda_{\{j\}} = \lambda_{\{i+j\}}$) [5,pp60-61].

Example 5. What is $(3\lambda_{\{5\}} + 5\lambda_{\{2\}}) \otimes (4\lambda_{\{3\}} - \lambda)$?

This would be calculated in the same way as the normal multiplication of two polynomials if the factorials of λ were instead powers. Thus, we would have $12\lambda_{\{8\}} - 3\lambda_{\{6\}} + 20\lambda_{\{5\}} - 5\lambda_{\{3\}}$. Remember that $\lambda_{\{1\}} = \frac{\lambda!}{(\lambda-1)!} = \lambda$, akin to regular powers.

6 Chromatic Polynomials of Composite Graphs

Some complex graphs can be considered to be some sort of composition of multiple graphs which would have simpler polynomials on their own. In this section, we can consider different ways of composing multiple graphs together and determine relations to define their chromatic polynomials easier.

Definition 7. Let G and H be graphs. The *disjoint union* of G and H, denoted by G $\cup$ H, is the graph defined with V(G $\cup$ H) = V(G) $\cup$ V(H) and E(G $\cup$ H) = E(G) $\cup$ E(H) [3,pXXI].

This would mean the disjoint union of the graphs would be just considering two graphs as one single entity. As no new edge is created between the two graphs, the new union is clearly not connected. Each subgraph of the union given would therefore be a component of the union, as defined below:

Definition 8. A *component* of a graph G is a connected subgraph H $\subset$ G such that there does not exist a connected subgraph H' $\subset$ G in which H $\subset$ H' [3,pXVIII].

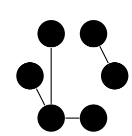
Theorem 5. *Let G and H be graphs with chromatic polynomials P(G, λ) and P(H, λ) respectively. Then the chromatic polynomial of the disjoint union of G and H is the product of the chromatic polynomials of G and H [5,p59], i.e.:*

$$P(G \cup H, \lambda) = P(G, \lambda)\, P(H, \lambda)$$

Proof. No edge connects any g $\in$ V(G) to any h $\in$ V(H), so G and H are two separate components of G $\cup$ H, so can be coloured independently of each other. Therefore the number of ways to colour the disjoint union is the product of the number of ways to colour each component [5,p59]. ■

A quick example of how this idea can be used would be for the chromatic polynomials of forests. Forests are basically trees which aren't necessarily connected, so a forest which isn't also a tree will have at least two components.

Example 6. The chromatic polynomial of $F_{(n,c)}$, which denotes any forest of order n and number of components, c, is the following [6]:



$$P(F_{(n,c)}, \lambda) = \lambda^c(\lambda - 1)^{n-c}$$

Proof. Let n = $\sum_{i=1}^{c} n_i$, so that n_i is the number of vertices in the ith component of $F_{(n,c)}$. $F_{(n,c)}$ is the disjoint union of its components, which are connected subgraphs with no cycles so each component is a tree. If we denote T_{n_i} as the ith component of $F_{(n,c)}$, we know from earlier that $P(T_{n_i}, \lambda) = \lambda(\lambda - 1)^{n_i - 1}$, so by the theorem of disjoint unions:

$$\begin{aligned} P(F_{(n,c)}, \lambda) &= \prod_{i=1}^{c} P(T_{n_i}, \lambda) \\ &= \lambda^c(\lambda - 1)^{n_1 + \ldots + n_c - 1 - \ldots - 1} \\ &= \lambda^c(\lambda - 1)^{n-c} \end{aligned}$$

■

Definition 9. Let G and H be graphs. The *join* of G and H, denoted by G + H, is the graph defined with V(G + H) = V(G) ∪ V(H) and E(G ∪ H) = E(G) ∪ E(H) ∪ E(I), where I is the complete bipartite graph with vertex partition sets I_1 and I_2, such that I_1 = V(G) and I_2 = V(H) [3,pXXI].

This would mean the join of two graphs would be constructed by taking the disjoint union of two graphs and adding an edge from each vertex in one component to every vertex in the other. So any vertex in G is adjacent to every vertex in H and vice versa.

Theorem 6. *Let G and H be graphs with chromatic polynomials P(G, λ) and P(H, λ) respectively. Then the chromatic polynomial of the join of G and H is the umbral product of the chromatic polynomials of G and H [5,pp60-61], i.e.:*

$$P(G + H, \lambda) = P(G, \lambda) \otimes P(H, \lambda)$$

Proof. Consider the chromatic reduction of both G and H to put both polynomials into factorial form. This means we have P(G, λ) = $\sum_{i=1}^{m} a_i \lambda_{\{i\}}$ and P(H, λ) = $\sum_{j=1}^{n} b_j \lambda_{\{j\}}$, where m and n are the orders of G and H respectively. The process is carried out by either contracting or adding an edge incident to pairs of non-adjacent vertices until we have two sums of the polynomials of complete graphs of varying orders. Applying the same reduction to G + H will reduce the G and H components in exactly the same way as if they were separate graphs, as G + H is defined so that any g ∈ V(G) is incident on all h ∈ V(H). Thus, the only pairs of vertices in G + H which aren't adjacent belong to the same subgraph that formed the join. Also, at every stage in the reduction, any vertex in V(G) will be adjacent to all vertices in V(H) and vice versa. This means we can consider the chromatic polynomial of G + H as the sum of all possible joins between a complete graph in the chromatic reduction of G and a complete graph in the chromatic reduction of H. However, by construction, the join of two complete graphs is also a complete graph, whose order is the sum of the orders of the two complete graphs joined. Thus, for each possible pair of terms taken from the reductions of the polynomials of G and H, one from each, there is a term in the join corresponding to the umbral product of those terms, i.e. for each pairing of a $\lambda_{\{a\}}$ term formed in the reduction of P(G,λ) and a $\lambda_{\{b\}}$ term in the reduction of P(H,λ), then there exists a corresponding term in the join graph P(G + H,λ) in the form of $\lambda_{\{a+b\}}$, unique before simplifying. Therefore, the chromatic polynomial of G + H can be found by multiplying the factorial forms of G and H's polynomials as if the factors were powers, also known as the umbral product. Hence, $P(G + H, \lambda) = P(G, \lambda) \otimes P(H, \lambda)$ [5,p61]. ■

Definition 10. Let G and H be graphs, such that $K_n \subset G$, $K_n \subset H$, so each graph has a complete subgraph of n vertices. Denote $G \cup_n H$ as a graph in which $G, H \subset G \cup_n H$ and $G \cap H = K_n$ [3,p7].

Theorem 7. *The chromatic polynomial of $G \cup_n H$ is given by the following relation [5,p59]:*

$$P(G \cup_n H, \lambda) = \frac{P(G,\lambda)\,P(H,\lambda)}{P(G \cap H, \lambda)} = \frac{P(G,\lambda)\,P(H,\lambda)}{P(K_n,\lambda)} = \frac{P(G,\lambda)\,P(H,\lambda)\,(\lambda - n)!}{\lambda!}$$

Proof. For any fixed colouring of G, H can be coloured in any number of ways as long as $G \cap H$ retains the same colouring as defined for G. Take any vertex v_1 in $G \cap H = K_n$. If one were to colour v_1, before considering a fixed colouring of G, it can be coloured in λ ways. For a set colouring of G, this vertex now has a fixed colour. Considering the kth vertex, $v_k \in K_n$, which could originally be coloured in (λ - k + 1) ways, since the vertex is adjacent to all (k - 1) other vertices already considered. Likewise, this now has only one fixed colouring. Thus, for a set colouring of G, the number of colourings of H is its chromatic polynomial divided by the original number of colourings for each vertex in the intersection, equal to $\lambda(\lambda-1)$ ... $(\lambda - n + 1) = \frac{\lambda!}{(\lambda-n)!} = P(K_n, \lambda)$. This means, given a set colouring for G:

$$P(H,\lambda) \to \frac{P(H,\lambda)}{P(K_n,\lambda)}$$

Since P(H, λ) is dependent on P(G, λ), then multiply the possible colourings of H given G's colouring by P(G, λ) to achieve $P(G \cup_n, \lambda) = \frac{P(G,\lambda)\,P(H,\lambda)}{P(G \cap H,\lambda)}$ [5,p59]. ■

Corollary 3. *Let G be the graph constructed from G_1 and G_2, in which the two subgraphs intersect at only one vertex. The chromatic polynomial of G is given as follows:*

$$P(G,\lambda) = \frac{P(G_1,\lambda)\,P(G_2,\lambda)}{\lambda}$$

The above corollary is evident in that a single vertex is effectively K_1, with the chromatic polynomial of λ.

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