MA241

Combinatorics Revision Guide

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Contents

Introduction

This revision guide for MA241 COMBINATORICS has been designed as an aid to revision, not a substitute for it. So practise, practise, PRACTISE, and good luck on the exam!

Disclaimer: Use at your own risk. Many proofs here are just rough ideas or omitted completely but this doesn't mean they aren't examinable!

Authors

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Based upon lectures given at the University of Warwick by Prof. Keith Ball.
Any corrections or improvements should be entered into our feedback form at `http://tinyurl.com/WMSGuides` (alternatively email revision.guides@warwickmaths.org).

History

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1 Basic Counting and the Binomial Theorem

Definition 1.1. A *set* is an unordered collection that can be described by listing its elements (e.g. $\{1, a, 321\}$), or specifying a common property.
We allow sets with repeated elements, or *multisets*, and write them in the form $\{\{x, x, y, y, z, z\}\}$ (for example). This multiset is distinct from $\{\{x, y, z\}\}$.

Definition 1.2. $\mathbb{N} = \{1, 2, 3, 4, \dots\}$ and $\mathbb{N}_0 = \{0\} \cup \mathbb{N} = \{0, 1, 2, 3, \dots\}$

Definition 1.3. The *cardinality* of a set A, written $|A|$, is the number of elements in A.

Example 1.4. $|\{\{x, x, y, y, z, z\}\}| = 6$, $|\{x, y, z\}| = 3$.

Definition 1.5. Elements in a tuple form a *list* − an ordered sequence.

Example 1.6. $(1, 3, 12)$, $(1, 12, 3)$ and $(1, 12, 3, 3)$ are lists, none of them are equal though.

Example 1.7. A *word* is a list where all elements come from a predefined set of *letters*, called the *alphabet*. We write the word as the sequence of its elements in order, i.e. (a, b, a, a, a, b) is written *abaaab*.

A word is *binary* if the alphabet is made up of two letters, so the word above is binary.

1.1 Counting lists and sets

There are two important rules for counting structures (i.e. anything) that are very useful in enumerative combinatorics:

Product rule If a structure can be constructed by making one of n_i independent choices at each step i (for k steps), then the total number of structures that can be made is $n_1 \cdot n_2 \cdot \dots \cdot n_k$.

Addition rule If a set of structures S can be partitioned into n disjoint sets S_i then $|S| = |S_1| + |S_2| + \dots + |S_n|$.

These rules can be used to counting the following things:

The number of k-lists with repetitions constructed from an n-set	n^k
The number of words of length k in an alphabet of n letters	n^k
The number of k-lists without repetitions constructed from an n-set is	$\frac{n!}{(n-k)!}$
The number of permutations of an n-set	$n!$
The number of subsets of an n-set	2^n
The number of binary words of length n	2^n

Note the equality between the last two entries above, you can prove this explicitly by assigning a binary word of length n to each subset of an n set.

1.2 Binomial Coefficients

Lemma 1.8 (Choosing subsets)**.** Let $n \in \mathbb{N}_0$ and $k \in \mathbb{N}_0$ such that $k \leq n$. Then the number of ways to choose a subset of k objects form among n is:

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

Theorem 1.9 (Binomial)**.** Let $x, y \in \mathbb{R}$ and $n \in \mathbb{N}_0$ then:

$$(x+y)^n = \sum_{k=0}^{n} (n)_k x^k y^{n-k}$$

Lemma 1.10 (Inductive property of Binomial Coefficients)**.** Let $n \in \mathbb{N}$ and $k \in \mathbb{N}$ such that $1 \leq k \leq n-1$ then:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Proof. This is a combinatorial proof, we look at the number of subsets of size k of the set $\{1, \dots, n\}$, by Lemma 1.8 this is $\binom{n}{k}$. To get the other side of the identity we take cases. The first case, we are looking at the number of subsets of size k of the set $\{1, \dots, n\}$ where we include 1 which is $\binom{n-1}{k-1}$. Now the second case is where we do not include 1 where we get $\binom{n-1}{k}$. Therefore by the addition rule we get the right hand side of the identity. □

2 Applications of Binomial Theorem

Theorem 2.1 (Orthogonality of Binomial Coefficients)**.** Let $r, n \in \mathbb{N}_0$ such that $r < n$ then:

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^r k^r = 0$$

Lemma 2.2. Let $n \in \mathbb{N}_0$ then:

$$\sum_{k=0}^{n} (-1)^k k^n = (-1)^n n!$$

Theorem 2.3 (Mean Value Theorem for Divided Differences)**.** Let $n \in \mathbb{N}_0$. if f is n times differentiable on an open interval containing $[0, n]$ then there exists $t \in (0, n)$ so that:

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k f(k) = (-1)^n f^{(n)}(t)$$

Theorem 2.4 (Multiset Formula)**.** Let $d \in \mathbb{N}_0$ and $m \in \mathbb{N}$, the number of multisets of size d with elements from a set of size m is:

$$\binom{d+m-1}{m-1} = \binom{d+m-1}{d}$$

Corollary 2.5 (Dimension of Spaces of Polynomials)**.** Let $d \in \mathbb{N}_0$ and $m \in \mathbb{N}$, the space of homogeneous polynomials of degree d in m variables has dimension:

$$\binom{d+m-1}{m-1}$$

The space of polynomials of degree at most d in m variables has dimension:

$$\binom{d+m}{m}$$

Definition 2.6. A collection of subsets $A_1, \dots, A_k$ of a set A is called a *partition* if the subsets are pairwise disjoint and their union is A. We can define an equivalence relation:
$R = \{(a, b) \mid \text{there is } A_i \text{ such that } a \in A_i \text{ and } b \in A_i\}$. Each subset is called an *equivalence class* of the relation.

Theorem 2.7. The number of ordered partitions of an n-set into k subsets of cardinalities $n_1, \dots, n_k$ is $\frac{n!}{n_1! n_2! \dots n_k!}$

Proof.

$$\frac{n!}{n_1!(n-n_1)!} \frac{(n-n_1)!}{n_2!(n-n_1-n_2)!} \cdots \frac{(n-n_1-\ldots-n_{k-1})!}{n_k!(n-n_1-\ldots-n_k)!} = \frac{n!}{n_1! n_2! \dots n_k!}$$

□

This is called a multinomial coefficient, and represented $\binom{n}{n_1, n_2, \dots, n_k}$. This would be the coefficient of $x_1^{n_1} x_2^{n_2} \dots x_k^{n_k}$ in $(x_1 + x_2 + \ldots + x_k)^n$.

Theorem 2.8 (Multinomial Theorem)**.** Let $x, y, z \in \mathbb{R}$ and $n \in \mathbb{N}_0$ then:

$$(x+y+z)^n = \sum_{i,j,k \geq 0, i+j+k=n} \frac{n!}{i!j!k!} x^i y^j z^k$$

Theorem 2.9 (Inclusion-Exclusion Formula)**.** For $A_1, A_2, \ldots, A_n$ and a natural $k \leq n$, denote

$$S_k = \sum_{i_1 < i_2 < \ldots < i_k} |A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k}|$$

Then $|A_1 \cup A_2 \cup \ldots \cup A_n| = S_1 - S_2 + S_3 - S_4 + \ldots + (-1)^{n-1} S_n$.

Proof. Let x belong to m sets. Then x contributes m to $S_1, \ldots, \binom{m}{i}$ to S_i. The total contribution is $\sum_{k=1}^{m} (-1)^{k-1} \binom{m}{k} = \sum_{k=0}^{m} (-1)^{k-1} \binom{m}{k} + \binom{m}{0} = 0 + 1 = 1$. □

Theorem 2.10. The number of surjections from a set of n elements to a set of k elements is

$$\sum_{i=0}^{k} (-1)^i \binom{k}{i} (k-i)^n$$

Proof. We use the inclusion-exclusion principle. Let A_i be the set of functions from A to B that never take on the value $i \in B$. Then $S_i = \sum_{j_1 < \ldots < j_i} |A_{j_1} \cap \ldots \cap A_{j_i}| = \binom{k}{i} |A_1 \cap \ldots \cap A_i| = \binom{k}{i} (k-i)^n$. So the total number is $\sum_{i=0}^{k} (-1)^i S_i = \sum_{i=0}^{k} (-1)^i \binom{k}{i} (k-i)^n$. □

3 The Fibonacci Numbers and Linear Difference Equations

Definition 3.1. The *Fibonacci numbers* are defined by the recursive relation

$$F_{n+2} = F_{n+1} + F_n$$

for $n \geq 0$ and with $F_0 = 1$, $F_1 = 1$.

Theorem 3.2. Here are some results for F_n:

1. $F_0 + F_1 + \cdots + F_n = F_{n+2} - 1$. (Proof by induction).
2. F_n is even $\iff$ n is a multiple of 3.
3. The number of domino tilings of a $2 \times (n-1)$ board is precisely F_n.
4. $$F_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^n.$$
5. $$F_n = \binom{n-1}{0} + \binom{n-2}{1} + \cdots + \binom{n - \frac{n+1}{2}}{\frac{n+1}{2} - 1}.$$

   This is the slanted diagonal on Pascal's triangle!

Theorem 3.3 (Fibonacci Matrix Theorem)**.** Let $n \in \mathbb{N}$, the powers of Q generate the Fibonacci numbers as follows:

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} u_{n+1} & u_n \\ u_n & u_{n-1} \end{pmatrix}$$

Theorem 3.4 (Divisibility of Fibonacci Numbers)**.** Let $m, n \in \mathbb{N}_0$, if $m|n$ then $u_m | u_n$.

Theorem 3.5 (Highest Common Factor of Fibonacci Numbers)**.** Let $m, n \in \mathbb{N}_0$, the highest common factor of u_m and u_n is the Fibonacci number u_h where $h = \text{hcf}\,(m, n)$.

4 Generating Functions and the Catalan Numbers

Definition 4.1. Let $a_0, a_1, \dots$ be an arbitrary sequence of numbers. The *generating function* for this sequence is the function

$$G(x) = \sum_{i \geq 0} a_i x^i$$

this is a *formal power series*, it does not need to converge or define a nice function! The sequence $\{a_i\}_{i \geq 0}$ is then called the coefficients of the generating function G.

If A, B are generating functions with coefficients $\{a_n\}$, $\{b_n\}$, AB is a generating function with coefficients $\{c_n\}$ defined as $c_n = \sum_{i+j=n} a_i b_j$.
The generating function for the Fibonacci numbers is:

$$0 + x + x^2 2x^3 + 3x^4 + 5x^5 + \cdots = \frac{x}{1 - x - x^2}$$

The generating function for the multiset formula is:

$$\sum_{d=0}^{\infty} \binom{d+m-1}{m-1} x^d = \frac{1}{(1-x)^m}$$

Definition 4.2. Given a regular polygon with labelled vertices a *triangulation* is a subdivision of the polygon into triangles obtained by drawing non-intersecting diagonals of a polygon.

Definition 4.3. The n-th *Catalan number*, C_n is defined by:
$C_n =$ the number of triangulations of a polygon with $n + 2$ sides.
E.g. $C_0 = 1$, $C_1 = 1$, $C_2 = 2$, $C_3 = 5$.

Catalan numbers can be used to count many things. Here we have Dyck paths as one example, in assignments there was another.

Theorem 4.4. Catalan numbers satisfy the following recursive relation,

$$C_{n+1} = C_0 \cdot C_n + C_1 \cdot C_{n-1} + \cdots + C_n \cdot C_0.$$

Proof. To prove this recursion we count the number of triangulations of an $(n + 3)$-gon in two ways.
Answer 1: (by definition) C_{n+1}.
Answer 2: First, label vertices of polygon from 1 to $n + 3$ clockwise and notice that the top edge of the polygon, between vertices $n+3$ and 1, is contained in exactly one triangle in each possible triangulation. To find the LHS of the equality we count the number of triangulations according to which triangle contains the top edge. In the general case, where the triangle K containing the top edge is defined by vertices $\{n+3, 1, n-k+2\}$, on the left of vertex $n-k+2$ we have C_k possible triangulations and on the right C_{n-k} possible triangulations. This gives a total of $C_k \cdot C_{n-k}$ possible triangulations for the case where the top edge is contained in triangle K. We now add over all the cases to, from $K = 1 to K = n$. This gives, $C_0 \cdot C_n + C_1 \cdot C_{n-1} + \cdots + C_n \cdot C_0$, as required. □

The following is a great way to practice finding the generating function of a sequence!

Theorem 4.5. Let $g(x) = \sum_{n=0}^{\infty} C_n x^n$, then $g(x) = \dfrac{1 - \sqrt{1-4x}}{2x}$.

Proof.

$$\begin{aligned} g(x) &= C_0 + C_1 x + C_2 x^2 + \cdots \\ (g(x))^2 &= (C_0 + C_1 X + C_2 x^2 + \cdots)(C_0 + C_1 x + C_2 x^2 + \cdots) \\ &= C_0^2 + (C_0 C_1 + C_1 C_0)x + (C_0 C_2 + C_1 C_1 + c_2 C_0)x^2 + (C_0 C_3 + C_1 C_2 + c_2 C_1 + C_3 C_0)x^3 + \cdots \end{aligned}$$

The coefficients of x^n now look like the recursion relation we proved in the last theorem so

$$\begin{aligned} (g(x))^2 &= C_1 + C_2 x + C_3 x^2 + \cdots = \frac{C_1 x + C_2 x^2 + C_3 x^3 + \cdots}{x} \\ &= \frac{1}{x}(-C_0 + (C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \cdots)) = \frac{-1 + g(x)}{x}. \end{aligned}$$

Rearranging this we get $(g(x))^2 - g(x) + 1$. If we solve this using the quadratic formula the result follows. □

Theorem 4.6 (Catalan Numbers)**.** Let $n \in \mathbb{N}$, then the number of ways to dissect a regular $(n+2)$-gon into n triangles using the diagonals is the Catalan number:

$$C_n = \frac{1}{n+1} \cdot \binom{2n}{n}$$

5 Permutation, Partitions and the Stirling Numbers

Definition 5.1. Let $n, k \in \mathbb{N}$, the Stirling number of the second kind $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$ is the number of partitions of a set of n symbols into k non-empty subsets.

Theorem 5.2 (Recurrence for Stirling II)**.** Let $n, k \in \mathbb{N}$, The Stirling numbers $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$ satisfy the following recurrence:

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\} = k \left\{ \begin{matrix} n-1 \\ k \end{matrix} \right\} + \left\{ \begin{matrix} n-1 \\ k-1 \end{matrix} \right\}$$

Theorem 5.3 (The Stirling numbers of The Second Kind)**.** Let $n \in \mathbb{N}$ and $k \in \mathbb{N}_0$ then:

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\} = \frac{1}{k!} \sum_{j=1}^{k} \binom{k}{j} (-1)^j (k-j)^n$$

Definition 5.4. Let $n \in \mathbb{N}_0$, the n-th Bell Number is the total number of partitions of a set of size n:

$$B_n = \sum_{k=0}^{n} \left\{ \begin{matrix} n \\ k \end{matrix} \right\}$$

Theorem 5.5 (The Bell Numbers)**.** Let $n \in \mathbb{N}_0$ The Bell numbers are given by:

$$B_n = e^{-1} \sum_{k=0}^{\infty} \frac{k^n}{k!}$$

Theorem 5.6 (Exponential Generating Function for Bell Numbers)**.**

$$\sum_{n=0} \frac{B_n}{n!} \cdot x^n = \exp\left(e^x - 1\right)$$

Definition 5.7. Let $n, k \in \mathbb{N}$, the unsigned Stirling number of the first kind $\left[\begin{matrix} n \\ k \end{matrix} \right]$ is the number of permutations of n symbols which are the product of k disjoint cycles.

Theorem 5.8 (Recurrence for Stirling I)**.** Let $n, k \in \mathbb{N}$, the Stirling number $\left[\begin{matrix} n \\ k \end{matrix} \right]$ satisfy the following recurrence:

$$\left[\begin{matrix} n \\ k \end{matrix} \right] = (n-1) \left[\begin{matrix} n-1 \\ k \end{matrix} \right] + \left[\begin{matrix} n-1 \\ k-1 \end{matrix} \right]$$

Theorem 5.9 (The Stirling Numbers of The First Kind)**.** Let $n\mathbb{N}$:

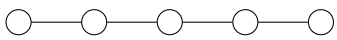
$$\sum_{k=1}^{n} \left[\begin{matrix} n \\ k \end{matrix} \right] x^k = x(x+1)(x+2)\cdots$$

6 Basic Graph Theory: Euler Trails and Circuits

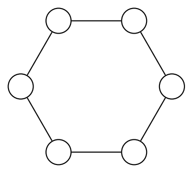
Definitions 6.1. 1. A graph G is a collection of *vertices* $V = \{v_1, v_2, \ldots, v_n\}$ together with a set of *edges* E each of which is a pair of vertices.

2. A graph G is said to be *connected* if there is a path in G between any pair of vertices. Each graph can be decomposed into connected components.
3. Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are *isomorphic* if there is a bijection ϕ from V_1 to V_2 with the property that $\{x, y\} \in E_1$ if and only if $\{\phi(x), \phi(y)\} \in E_2$.
4. If $G = (V, E)$ is a graph then a *subgraph* of G is a graph (V', E') where $V' \subset V$ and $E' \subset E$ (and the elements of E' are pairs of elements of V').
5. An *induced* subgraph of (V, E) is a graph (V', E') where $V' \subset V$ and E' consists of all pairs in E that are subsets of V': we include all the edges of G that we can.
6. A *walk* in a graph is a sequence of vertices, each one adjacent to the next, possibly with repetition. It is *closed* if its first and last vertices are the same.
7. A *path* is a walk which uses distinct vertices. A *cycle* is a closed walk which uses distinct vertices except at the ends.
8. A graph is *bipartite* if its vertex set can be partitioned into two parts A and B in such a way that all edges cross from A to B: (none is inside either part).
9. A *complete* graph is one where every vertex is connected via an edge to every other vertex. We use K_n to denote the complete graph on n vertices.

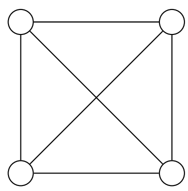
Example 6.2. The following families of graphs are good to know:



P_n – the path on n vertices



C_n – the cycle on n vertices



K_n – the complete graph on n vertices

Theorem 6.3 (Characterisation of Bipartite Graphs)**.** A graph is bipartite if and only if it contains no odd cycles.

Lemma 6.4 (Odd Walk Lemma)**.** If a graph contains a closed walk of odd length then it contains a cycle of odd length.

Definition 6.5. A walk in which all the edges are distinct is called a *trail*, if it is closed it is called a *circuit*.

Lemma 6.6 (Handshaking Lemma)**.** The number of vertices of odd degree in a graph is even.

Proof. Let $G = (V, E)$ be a graph with m edges. For each edge there are corresponding vertices v_1 and v_2 for which this edge contributes 1 to $\deg(v_1)$ and 1 to $\deg(v_2)$. So we have:

$$\sum_{v \in V} \deg(v) = 2m.$$

The result follows since if we had an odd number of vertices of odd degree then the sum would be odd which would contradict the above formula. □

Theorem 6.7 (Euler Circuits)**.** A connected graph G has an *Euler trail* if and only if it has just two vertices of odd degree, and an *Euler circuit* if and only if it has none.

To prove this we assume we have a closed Eulerian trail of maximal length and argue that if it does not contain all edges we can extend it to contain more edges, contradicting our assumption.

7 Trees, Spanning Trees and Cycles

Definition 7.1. A *tree* is a graph that is connected but contains no cycle.

Lemma 7.2. Let G be a graph, the following are equivalent:
1. G is a tree
2. G is a maximal graph with no cycles, (i.e. maximal in the sense that G is a graph where adding one more edge e would mean $G + e$ contains a cycle)
3. G is a minimal connected graphs, (i.e. minimal in the sense that G is a graph where removing one edge e would mean $G - e$ is not connected)

Definition 7.3. A *spanning tree* of a graph G is a tree in G that connects all the vertices of G.

Lemma 7.4 (Spanning Trees)**.** Every connected graph contains a spanning tree.

Theorem 7.5. Let $n \in \mathbb{N}$ and G be a connected graph with n vertices, the following are equivalent:
1. G is a tree
2. G has exactly $n - 1$ edges
3. G contains no cycles
4. Every pair of vertices is connected by a unique path

Proof. (1) $\Rightarrow$ (2) Proof by induction on n. Suppose we have a tree with 1 vertex, it is a tree with zero edges. Assume the statement is true for any trees with vertices strictly less than n. Let G_1, G_2 be trees with k and $n - k$ vertices respectively with $k < n$ such that G can be made by joining G_1 and G_2 by an edge. By induction assumption, G_1 has $k - 1$ edges and G_2 has $n - k - 1$ edges. So the number of edges in G is $k - 1 + n - k - 1 + 1 = n - 1$. Therefore true by induction
(2) $\Rightarrow$ (1) Proof by contradiction, suppose G has $n - 1$ edges, is connected and is not a tree. Then by Lemma 7.2 we can remove an edge and still get something that is connected. This is a contradiction as we cannot have a connected graph with n vertices and $n - 2$ edges. Hence, G is a tree.
(1) $\Rightarrow$ (3) Poof by contrapositive, Suppose G has a cycle, we can remove any of its edges and still get a connected graph then G cannot be a tree.
(3) $\Rightarrow$ (4) Proof by contradiction, suppose G has no cycles. Suppose 2 vertices are joined by two distinct paths, then we have a cycle which is a contradiction.
(4) $\Rightarrow$ (1) Proof by contradiction, suppose ever pair of vertices is connected by a unique path and that G is not a tree. By Lemma 7.2 we could remove an edge and get something that is connected. This means there are two ways of getting from one vertex to another which is a contradiction. □

Definition 7.6. Let $n \in \mathbb{N}$, if G is a graph on the vertices $i = 1, 2, \ldots, n$ and for each i the vertex i has degree d_i, then the *Laplacian* of G is the symmetric matrix (a_{ij}) given by:

$$a_{ij} = \begin{cases} d_i & \text{if } i = j \\ -1 & \text{if } ij \text{ is an edge} \\ 0 & \text{if } i \neq j \text{ and } ij \text{ is not an edge} \end{cases}$$

Theorem 7.7 (Kirchoff's Matrix Tree Theorem)**.** Let L be the Laplacian of a graph G then the number of spanning trees of G is any $(n-1) \times (n-1)$ principal minor of L.

Theorem 7.8 (Cayley's Formula)**.** Let $n \in \mathbb{N}$ such that $n \geq 2$, then there are n^{n-2} trees on n vertices.

Suppose we choose a collection of $n-1$ of the e_{ij} vectors and put them side by side to form a matrix: then the determinants of the $(n-1) \times (n-1)$ submatrices all have the same size. The $(n-1) \times (n-1)$ minors are zero if the edges form a cycle but are ± 1 if they form a tree.

Suppose we are given a graph G with m edges and we form the $n \times m$ incidence matrix $\tilde{B}$ using the edge vectors. Then the number of spanning trees of G is the sum of the squares of the $(n-1) \times (n-1)$ determinants of the incidence matrix with a row deleted.

Theorem 7.9 (Cauchy-Binet)**.** Let $k, m \in \mathbb{N}$ such that $k \leq m$ and B be a $k \times m$ matrix. The sum of the squares of the $k \times k$ minors of B is:

$$\det\left(B \cdot B^T\right)$$

If G is a graph on n vertices with m edges form the $n \times m$ matrix $\tilde{B}$ whose columns are the vectors e_{ij} corresponding to edges in G. If L is the Laplacian of G then:

$$\tilde{B}.\tilde{B}^T = L$$

8 Hall's Theorem

Definition 8.1. If G is a bipartite graph with vertex sets A and B then a *complete matching* from A into B is a set of disjoint edges which cover the vertices of A. (One edge coming out of each vertex of A).

Theorem 8.2 (Hall's Marriage Theorem)**.** Let G be a bipartite graph with vertex classes A and B. For each subset $U \subset A$ let $\Gamma(U)$ be the set of neighbours of vertices in U:

$$\Gamma(U) = \{b : ab \text{ is an edge for some } a \in U\}$$

If for every $U \subset A$ the set $\Gamma(U)$ is at least as large as U then G contains a complete matching from A into B.

9 Ramsey Theory

Another important idea used in a number of combinatorial arguments is the *pigeonhole principle* which states that: If $n+1$ objects are placed in n boxes then at least one box has two or more objects in it.

There is a more general version of this, known as the *stronger pigeonhole principle* which states that: If $n(r-1)+1$ objects are placed in n boxes then at least one box has r or more objects in it.

This can be used to prove statements such as the following:

Theorem 9.1 (Erös-Szekeres)**.** Every sequence of real numbers $a_1, \ldots, a_{n^2+1}$ contains an increasing or decreasing subsequence of length $n+1$.

Proof. Suppose there does not an increasing subsequence of length $n+1$. Let m_k be the length of the largest increasing subsequence starting at a_k. We have that m_k satisfy $1 \leq m_k \leq n$ and we have n^2+1 different lengths m_k, by the strong pigeonhole principle at least $n+1$ of them much be equal. Suppose $m_{i_1} = m_{i_2} = \cdots = m_{i_{n+1}}$ then $a_{i_1} > a_{i_2} > \cdots > a_{i_{n+1}}$ therefore this is a decreasing subsequence of length $n+1$. □

Definition 9.2. Let $s,t \in \mathbb{N}$ such that $s,t \geq 2$, we set $R(s,t)$ to be the least number n so that no matter how we 2-colour the edges of the complete graph K_n then we find either a red K_s or a blue K_t.

Theorem 9.3. In a party in 6 or more people, there is a group of 3 who are either mutual strangers or mutual acquaintances i.e. $R(3,3) = 6$

Proof. Here we show that $R(3,3) \leq 6$, it is an exercise to show $R(3,3) > 5$. We think of people as vertices in a complete graph on 6 vertices where edges are coloured either blue or red according to whether they are strangers or acquaintances. We claim no matter how the edges are coloured, there is always a monochromatic triangle i.e. all edges in the triangle are one colour. Pick any vertex, by the strong pigeonhole principle at least 3 of the 5 edges must be the same colour. Suppose this is red, there are 3 other edges joining the vertices in our monochromatic triangle. (Draw a picture). Proceeding by cases either one of these edges is red which implies there exists a monochromatic triangle. Or there are no reds in which case we have a monochromatic blue triangle. □

Lemma 9.4. Let $s \in \mathbb{N}$ such that $s \geq 2$ then $R(s,2) = s$.

Theorem 9.5 (Ramsey Recurrence)**.** Let $s,t \in \mathbb{N}$ such that $s,t \geq 3$ then:

$$R(s,t) \leq R(s-1,t) + R(s,t-1)$$

Theorem 9.6 (Ramsey Bound)**.** Let $s, t \in \mathbb{N}$ such that $s, t \geq 3$ then:

$$R(s,t) \leq \binom{s+t-2}{s-1}$$

Lemma 9.7 (Universal Graphs for Trees)**.** Let $t \in \mathbb{N}$, if G is a graph in which every vertex has degree at least $t-1$ then it contains a copy of every tree of order t.

Theorem 9.8 (Ramsey for a Complete Graph Against Trees)**.** Let $s, t \in \mathbb{N}$ such that $s, t \geq 2$ and $n = (s-1)(t-1)+1$. If you 2-colour K_n then you find either a red K_s or a blue copy of every tree of order t.

Theorem 9.9 (Erdös Lower Bound for $R(s, s)$)**.** Let $s \in \mathbb{N}$ such that $s \geq 3$, then:

$$R(s,s) \geq 2^{(s-1)/2}$$

10 Planar graph

Definition 10.1. A graph is *planar* if it can be drawn on the plane in such a way that no two edges cross each other.
Such a drawing of a graph G is called a *planar representation* of G. A *face* (or *region*) is an area of the plane bound by edges in a planar representation.

Theorem 10.2 (Euler's Formula)**.** Let G be a connected planar graph with v vertices, e edges and f faces, then:

$$f + v = e + 2$$

Proof. Suppose G is a graph with v vertices, e edges and f faces. Remove edges of G until it is minimally connected and call this G'. G' a tree which has to have 1 face otherwise there is a cycle and the graph would then not be a tree and hence no minimally connected. So if it has $v' = v$ vertices then it has $e' = v - 1$ edges and $f' = 1$ face so $v' - e' + f' = 2$. Add back the edges that were removed one by 1, every time we add an edge, we increase the number of edges by one and increase the number of faces by 1. Therefore $v - e + r = 2$. ☐

Lemma 10.3 (Maximal Planar Graphs)**.** Let $n \in \mathbb{N}$, a maximal planar graph on $n \geq 3$ vertices has $3n - 6$ edges.

Proof. Let G be planar graph and $r_1, \ldots, r_f$ the number of edges bounding each face in planar representation of G. Each edge bounds 2 faces so $2e = r_1 + r_2 + \cdot + r_f$. Each r_i is at least 3 so $2e \geq 3f$ hence $f \leq \frac{2}{3}e$. Substituting this in the Euler's formula, we get $2 = v - e + f \leq v - e + \frac{2}{3}e$. ☐

Theorem 10.4 (The 5 Colour Theorem)**.** Every planar graph can be coloured with 5 colours.

The proof of this theorem is by induction on the number of vertices. The difficult case being if we have every vertex of degree 5 or more (there must be one of at most five from earlier).

Theorem 10.5 (Non-Planar Graphs)**.** The graphs K_5 and $K_{3,3}$ are not planar.

Theorem 10.6 (Kuratowski's Theorem)**.** A graph fails to be planar if and only if it contains a subdivision of K_5 or $K_{3,3}$.

11 Boolean Functions

Definition 11.1. Let $n \in \mathbb{N}$, a *Boolean* function is a function from $\{0,1\}^n \to \{0,1\}$. To each input sequence $(x_1, x_2, \ldots, x_n)$ of bits, it assigns an output bit.

Example 11.2.

$(0,0)$	0	1	0	0
$(0,1)$	0	1	0	1
$(1,0)$	1	0	0	1
$(1,1)$	1	0	1	1
(X,Y)	X	$\overline{X}$	$X \wedge Y$	$X \vee Y$

Definitions 11.3.
1. A disjunction of elementary conjunctions is called a *disjunctive normal form* or *DNF*.
2. A conjunction of elementary disjunctions is called a *conjunctive normal form* or *CNF*.

Lemma 11.4. Each function can be written in DNF or CNF.

It is easy to check the satisifiability of a DNF.

Definition 11.5. *NP* is the space of problems for which we can check the validity of a solution in polynomial time.

Lemma 11.6. If a problem is in NP and is NP-hard then it is NP-complete.

Lemma 11.7. Satisfiability for CNF's with 3 symbols in each disjunction is NP-complete.

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