MA244

Analysis 3 - Definitions and Theorems Revision Guide

Written by Sean Middlehurst

Contents

Introduction

This revision guide for MA244 Analysis III has been designed as an aid to revision, not a substitute for it. This guide is useful for revising through key definitions, theorems and some shorter proofs found in the course. However, a lot of the calculation methods and practical applications of the content of this module are omitted, in which it would be best to refer to the lectures and the online notes for said techniques.

Disclaimer: Use at your own risk. No guarantee is made that this revision guide is accurate or complete, or that it will improve your exam performance.

Authors

Written by Sean Middlehurst.
Based upon the lecture notes for MA244 Analysis III, written by Jose Rodrigo at the University of Warwick.
Any corrections or improvements should be reported by email to comms@warwickmaths.org.

1 Riemann Integral

1.1 Defining the Integral

Definition 1.1. Let I be a nonempty, closed interval in $\mathbb{R}$. A *partition* of I is a collection of nonempty intervals $\{I_1, ..., I_n\}$ of almost-disjoint, nonempty closed intervals whose union is I.

In practice, if we let $I = [a, b]$, then a partition would be determined by points $\{x_i\}$ such that:

$$a = x_0 < x_1 < ... < x_{n-1} < x_n = b$$

In which I_J corresponds with the interval $[x_{j-1}, x_j]$.

Definition 1.2. For a partition P of an interval I, we denote:

$$M = \sup_I f \qquad m = \inf_I f \qquad M_k = \sup_{I_k} f \qquad m_k = \inf_{I_k} f$$

Definition 1.3. Given $f : [a, b] \to \mathbb{R}$ and a partition P of $[a, b]$ we define the *upper Riemann sum* of f with respect to P as:

$$U(f, P) := \sum_{k=1}^{n} M_k |I_k|$$

and the *lower Riemann sum* of f with respect to P as:

$$L(f, P) := \sum_{k=1}^{n} m_k |I_k|$$

Definition 1.4. Given $f : [a, b] \to \mathbb{R}$ bounded, we define the *upper Riemann integral* of f as:

$$U(f) := \inf_{P \in \mathcal{P}} U(f, P)$$

and the *lower Riemann integral* of f as:

$$L(f) := \sup_{P \in \mathcal{P}} L(f, P)$$

Where we define $\mathcal{P}$ as the set of all partitions of $[a, b]$.

Definition 1.5. Given $f : [a, b] \to \mathbb{R}$ bounded, we say that it is *Riemann integrable* if and only if $L(f) = U(f)$, and define its Riemann integral as:

$$\int_a^b f(x)dx := L(f) = U(f)$$

Definition 1.6. A partition $Q = \{J_1, ..., J_l\}$ of $[a, b]$ is a *refinement* of a partition $P = \{I_1, ..., I_n\}$ if every interval I_k in P is the union of one or more intervals J_k from the partition Q.

Theorem 1.7. Let $f : [a, b] \to \mathbb{R}$ be a bounded function and P and Q be partitions of $[a, b]$, with Q being a refinement of P. Then:

$$L(f, P) \leq L(f, Q) \leq U(f, Q) \leq U(f, P)$$

Theorem 1.8. Let $f : [a, b] \to \mathbb{R}$ be a bounded function and P, Q two partitions of $[a, b]$. Then:

$$L(f, P) \leq U(f, Q)$$

Proof. We set R to be a refinement of P and Q, by taking the union of all endpoints of both P and Q. By the previous theorem applied to P, Q and R, we obtain:

$$L(f, P) \leq L(f, R) \leq U(f, R) \leq U(f, Q)$$

□

Corollary 1.9. Given $f : [a, b] \to \mathbb{R}$ bounded, we have:

$$L(f) \le U(f)$$

Theorem 1.10. Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Then f is integrable if and only if for $\forall\, \varepsilon > 0, \exists$ a partition P of $[a, b]$ such that:

$$U(f, P) - L(f, P) < \varepsilon$$

Theorem 1.11. Let $f : [a, b] \to \mathbb{R}$ be a bounded function. f is integrable if and only if $\exists\, P_n$, a sequence of partitions such that:

$$\lim_{n\to\infty} U(f, P_n) - L(f, P_n) = 0$$

Definition 1.12. Given $f : [a, b] \to \mathbb{R}$, we say that f is *continuous* at x if $\forall\, \varepsilon > 0\, \exists\, \delta > 0$ such that:

$$y \in [a, b] \text{ and } |x - y| < \delta \implies |f(y) - f(x)| < \varepsilon$$

Definition 1.13. Given $f : [a, b] \to \mathbb{R}$, we say that f is *uniformly continuous* if $\forall\, \varepsilon > 0\, \exists\, \delta > 0$ such that:

$$x, y \in [a, b] \text{ and } |x - y| < \delta \implies |f(y) - f(x)| < \varepsilon$$

Theorem 1.14. Let $f : [a, b] \to \mathbb{R}$. f is continuous implies f is uniformly continuous.

Theorem 1.15. Let $f : [a, b] \to \mathbb{R}$ be a monotonic function. Then it is Riemann integrable.

1.2 Properties of the Integral

Theorem 1.16. Let $f, g : [a, b] \to \mathbb{R}$ be Riemann integrable functions, and $c \in \mathbb{R}$. Then $f + g, cf$ are Riemann integrable functions, such that:

$$\int_a^b cf = c\int_a^b f \qquad \int_a^b (f + g) = \int_a^b f + \int_a^b g$$

Theorem 1.17. Let $f, g : [a, b] \to \mathbb{R}$ be Riemann integrable functions such that $f \le g$. Then:

$$\int_a^b f \le \int_a^b g$$

Note: We can now see that if $f = 0$, then clearly $g \ge 0 \implies \int g \ge 0$.

Corollary 1.18. Let $f : [a, b] \to \mathbb{R}$ be integrable. Let $m = \inf f$ and $M = \sup f$. Then:

$$m(b - a) \le \int_a^b f \le M(b - a)$$

Corollary 1.19. Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then $\exists\, c \in [a, b]$ such that:

$$f(c) = \frac{1}{b - a}\int_a^b f$$

Note: We use the value $\frac{1}{b-a}\int_a^b f$ to correspond to the average of f. Replacing f with the constant given by $\frac{1}{b-a}\int_a^b f$ gives the same value for the integral for both functions.

Theorem 1.20. Let $f : [a, b] \to \mathbb{R}$ be an integrable function. Then $|f|$ is integrable, such that:

$$|\int_a^b f| \le \int_a^b |f|$$

Theorem 1.21. Let $f : [a, b] \to \mathbb{R}, c \in (a, b)$. Then f is Riemann integrable on $[a, b]$ if and only if it is Riemann integrable on $[a, c]$ and $[c, b]$. Moreover:

$$\int_a^c f + \int_c^b f = \int_a^b f$$

Theorem 1.22. Let $f : [a, b] \to \mathbb{R}$ be a bounded, Riemann integrable function and $\phi : \mathbb{R} \to \mathbb{R}$ a continuous function. Then $\phi \circ f$ is Riemann integrable.

Note: The composition of two Riemann integrable functions is not necessarily integrable.

Theorem 1.23. Let $f, g : [a, b] \to \mathbb{R}$ be Riemann integrable functions. Then the product fg is Riemann integrable. If in addition $\frac{1}{g}$ is bounded, then $\frac{f}{g}$ is Riemann integrable.

1.3 The Fundamental Theorem of Calculus

Theorem 1.24. Let $F : [a, b] \to \mathbb{R}$ be a continuous function that is differentiable on (a, b) with $F' = f$. Assume that $f : [a, b] \to \mathbb{R}$ is an integrable function. Then:

$$\int_a^b f(x)dx = F(b) - F(a)$$

Theorem 1.25. Let $f : [a, b] \to \mathbb{R}$ be an integrable function and define the function $F : [a, b] \to \mathbb{R}$ by:

$$F(x) := \int_a^x f(t)dt$$

Then F is continuous on $[a, b]$. If f is continuous at $c \in [a, b]$ then $F'(c) = f(c)$.

Theorem 1.26. Let $f : [a, b] \to \mathbb{R}$ be an integrable function on $[a, b]$ and continuous (from the right) at a. Then:

$$\lim_{h \to 0} \frac{1}{h} \int_a^{a+h} f(t)dt = f(a)$$

Similarly, if f is continuous (from the left) at b:

$$\lim_{h \to 0} \frac{1}{h} \int_{b-h}^b f(t)dt = f(b)$$

We can also consider a family of intervals I_h such that $x \in I_h, |I_h| \to 0$ and (assuming f is continuous at x):

$$\lim_{h \to 0} \frac{1}{|I_h|} \int_{I_h} f(t)dt = f(x)$$

Theorem 1.27. Let $f, g : [a, b] \to \mathbb{R}$ be continuous functions on $[a, b]$ that are differentiable on (a, b), and such that f' and g' are integrable on $[a, b]$. Then:

$$\int_a^b f(x)g'(x)dx = f(b)g(b) - f(a)g(a) - \int_a^b f'(x)g(x)dx$$

Theorem 1.28. Let $f : [a, b] \to \mathbb{R}$ be a differentiable function such that f' is integrable on $[a, b]$. Let g be a continuous function on $f([a, b])$, the image of $[a, b]$ under the map f. Then:

$$\int_a^b g(f(x))f'(x)dx = \int_{f(a)}^{f(b)} g(t)dt$$

1.4 Improper Integrals

Definition 1.29. Let $f : [a, b] \to \mathbb{R}$ be a Riemann integrable function for every $[c, b]$ with $a < c$. Then the *improper integral* of f on $[a, b]$ is defined as:

$$\int_a^b f(x)dx = \lim_{\varepsilon \to 0^+} \int_{a+\varepsilon}^b f(x)dx$$

Similarly, for a function Riemann integrable for every $[a, c]$ with $c < b$, we define:

$$\int_a^b f(x)dx = \lim_{\varepsilon \to 0^+} \int_a^{b-\varepsilon} f(x)dx$$

If the limit is finite, the integral is said to converge, and diverge otherwise.

Definition 1.30. Let $f : [a, b] \to \mathbb{R}$ be a function that is integrable on any closed interval not containing $c \in [a, b]$, that is on all $[a, c - \varepsilon]$ and $[c + \delta]$, for $\varepsilon, \delta > 0$ sufficiently small. Then we define the *improper integral* of f on $[a, b]$ as:

$$\int_a^b f(x)dx = \lim_{\varepsilon \to 0^+} \int_a^{c-\varepsilon} f(x)dx + \lim_{\delta \to 0^+} \int_{c+\delta}^b f(x)dx$$

Definition 1.31. Let $f : [a, \infty) \to \mathbb{R}$ be an integrable function for every $[a, y]$. Then the *improper integral* of f on $[a, \infty]$ is defined as:

$$\int_a^\infty f(x)dx = \lim_{y \to \infty} \int_a^y f(x)dx$$

Similarly, for a function $g : (-\infty, b] \to \mathbb{R}$ is integrable for every $[y, b]$, we define:

$$\int_{-\infty}^b g(x)dx = \lim_{y \to -\infty} \int_y^b g(x)dx$$

Definition 1.32. Let $f : \mathbb{R} \to \mathbb{R}$ be a function that is integrable on any bounded interval $[a, b]$. Then we define the *improper integral* of f on $[a, b]$ as:

$$\int_{-\infty}^\infty f(x)dx = \lim_{a \to \infty} \int_a^c f(x)dx + \lim_{b \to \infty} \int_c^b f(x)dx$$

where $c \in \mathbb{R}$.

Note: Improper integrals form a linear space, i.e., if f, g are improperly integral on the same domain, $\alpha f + \beta g$ is also improperly integrable $\forall\, \alpha, \beta \in \mathbb{R}$.

1.5 The Cantor Set

Definition 1.33. The *Cantor Set* is defined as follows:

- Set $C_0 = [0, 1]$.
- Set C_1 by removing the open interval $(\frac{1}{3}, \frac{2}{3})$ from C_0, meaning $C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$.
- Inductively, given C_k, set C_{k+1} by taking each closed interval in C_k and removing the open middle third interval, of length $\frac{1}{3^{k+1}}$, from it.
- The Cantor Set, C is given as the limit of these sets, i.e.:

$$C := \lim C_k = \bigcap_{k=1}^{\infty} C_k$$

Definition 1.34. The *Devil's Staircase* function is the function defined as follows:

- Set $f_0(x) = x$.
- Set f_1 as follows:

$$f_1(x) := \begin{cases} \frac{3}{2}x & x \in [0, \frac{1}{3}] \\ \frac{1}{2} & x \in [\frac{1}{3}, \frac{2}{3}] \\ \frac{1}{2} + \frac{3}{2}(x - \frac{2}{3}) & x \in [\frac{2}{3}, 1] \end{cases}$$

- Inductively, given f_k, set f_{k+1} as follows:

$$f_n(x) := \begin{cases} \frac{1}{2}f_n(3x) & x \in [0, \frac{1}{3}) \\ \frac{1}{2} & x \in [\frac{1}{3}, \frac{2}{3}) \\ \frac{1}{2} + \frac{1}{2}f_n(3x - 2) & x \in [\frac{2}{3}, 1] \end{cases}$$

Theorem 1.35. The limit of the sequence (f_n) exists and is continuous with $f(0) = 0$ and $f(1) = 1$.

2 Sequences and Series of Functions

2.1 Pointwise and Uniform Convergence

Definition 2.1. Let $(f_n)_{n=1}^{\infty}$ be a sequence of functions, with $f_n : \Omega \to \mathbb{R}$. We say that (f_n) or f_n *converges pointwise* to $f : \Omega \to \mathbb{R}$ if and only if for every $x \in \Omega$, we have $\lim_{n\to\infty} f_n(x) = f(x)$. Pointwise convergence is denoted by $f_n \to f$.

Note: Pointwise limits of sequences of continuous functions need not be continuous.

Definition 2.2. Let $f_n : \Omega \to \mathbb{R}$ be a sequence of functions. We say that (f_n) converges uniformly to $f : \Omega \to \mathbb{R}$ if and only if $\forall\, \varepsilon > 0\, \exists\, N \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \varepsilon\, \forall\, x \in \Omega$ and $\forall\, n > N$. Uniform convergence is denoted by $f_n \rightrightarrows f$.

Definition 2.3. We define the following norm:

$$\|f\|_\infty = \sup_{x\in\Omega} |f(x)|$$

As a norm, $\|\cdot\|_{infty}$ holds the following properties:

- $\|f\|_\infty \geq 0$,
- $\|\lambda\|_\infty\| = |\lambda|\|f\|_\infty, \forall\lambda \in \mathbb{R}$,
- $|f + g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$.

Note: From this notation, we have:

$$f_n \rightrightarrows f \iff \forall\, \varepsilon > 0, \exists\, N \text{ such that } \|f_n - f\|_\infty < \varepsilon\, \forall\, n > N$$

Definition 2.4. A sequence (f_n) of functions in Ω is called *uniformly Cauchy* if and only if $\forall\, \epsilon > 0\, \exists\, N \in \mathbb{N}$ such that $\|f_n - f_m\|_\infty\, \forall\, n, m > N$.

Theorem 2.5. A sequence (f_n) is uniformly convergent if and only if it is uniformly Cauchy.

Theorem 2.6. Let (f_n) be a sequence of continuous functions in Ω that converges uniformly to $f : \Omega \to \mathbb{R}$. Then f is continuous.

Theorem 2.7. $(C_b; \|\cdot\|_\infty)$ is a complete space, i.e., every Cauchy sequence converges to a continuous bounded function.

Theorem 2.8. Let $(f_n), f : [a, b] \to \mathbb{R}$ be Riemann integrable functions that converges uniformly to $f : [a, b] \to \mathbb{R}$. Then f is Riemann integrable and $\int f_n \to \int f$.

Definition 2.9. Given $f : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$, we say that f is continuous at x if $\forall\, \varepsilon > 0 \,\exists\, \delta > 0$ such that:

$$y \in \Omega \text{ and } |x - y| < \delta \implies |f(y) - f(x)| < \varepsilon$$

Theorem 2.10. Given $f : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$, we say that it is *uniformly continuous* if $\forall\, \varepsilon > 0 \,\exists\, \delta > 0$ such that:

$$x, y \in \Omega \text{ and } |x - y| < \delta \implies |f(y) - f(x)| < \varepsilon$$

Theorem 2.11. Let $f : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ be a continuous function. Assume that Ω is closed and bounded. Then it is uniformly continuous.

Theorem 2.12. Let $f : [a, b] \times [c, d] \to \mathbb{R}$ be a continuous function. Define:

$$I(t) := \int_a^b f(x, t)dx$$

is a continuous function in $[c, d]$.

Theorem 2.13. Let f, $\frac{\partial f}{\partial t}$ be continuous functions on $[a, b] \times [c, d]$. Then, for $t \in (c, d)$:

$$\frac{d}{dt} \int_a^b f(x, t)dx = \int_a^b \frac{\partial f}{\partial t}(x, t)dx$$

Theorem 2.14. Let $f : [a, b] \times [c, d] \to \mathbb{R}$ be a continuous function. Then:

$$\int_a^b \left(\int_c^d f(x, y)dy \right) dx = \int_c^d \left(\int_a^d f(x, y)dx \right) dy$$

Theorem 2.15. Let (f_n) be a sequence of C^1 functions on $[a, b]$. Assume $f_n \to f$ pointwise and that f_n' converges uniformly to g. Then f is C^1 and $g = f'$ or $f_n' \to f'$.

2.2 Series of Functions

Definition 2.16. Let (f_k) be a sequence of functions $f_k : \Omega \to \mathbb{R}$. Let (S_n) be the *sequence of partial sums*, with $S_n : \Omega \to \mathbb{R}$ defined by:

$$S_n(x) := \sum_{k=1}^{n} f_k(x)$$

Then the series:

$$\sum_{k=1}^{\infty} f_k(x)$$

converges pointwise to $S : \Omega \to \mathbb{R}$ on Ω if $S_n \to S$ pointwise on Ω and uniformly to S on Ω if $S_n \rightrightarrows S$ uniformly on Ω.

Theorem 2.17. Let (f_k), with $f_k : [a, b] \to \mathbb{R}$, be a sequence of integrable functions. Assume that $S_n = \sum_{k=1}^{n} f_k$ converges uniformly. Then $\sum_{k=1}^{\infty} f_k$ is Riemann integrable and:

$$\int \sum_{k=1}^{\infty} f_k = \sum_{k=1}^{\infty} \int f_k$$

Theorem 2.18. Let (f_k), with $f_k : [a, b] \to \mathbb{R}$, be a sequence of C_1 functions such that $S_n = \sum_{k=1}^{n} f_k$ converges pointwise. Assume that $\sum_{k=1}^{n} f_k'$ converges uniformly. Then:

$$\left(\sum_{k=1}^{\infty} f_k(x) \right)' = \sum_{k=1}^{\infty} f_k'(x)$$

Theorem 2.19. The Weierstrass M-test: Let (f_k) be a sequence of functions $f_n : \Omega \to \mathbb{R}$, and assume that $\forall\, k \,\exists\, M_k > 0$ such that $|f_k(x)| \leq M_k \,\forall\, x \in \Omega$ and $\sum_{k=1}^{\infty} M_k < \infty$. Then $\sum_{k=1}^{\infty} f_k$ converges uniformly on Ω.

2.3 Absolute Continuity

Definition 2.20. Let I be an interval in $\mathbb{R}$. A function $f : I \to \mathbb{R}$ is *increasing* if $f(x) \leq f(y)$ whenever $x, y \in I, x < y$ and similarly, is *decreasing* if $f(x) \geq f(y)$ for the same x, y. A function is *monotone* if it is either increasing or decreasing. A function is *strictly increasing/decreasing/monotone* if we replace the inequalities in the previous definition with strict inequalities.

Theorem 2.21. Let $f : [a, b] \to \mathbb{R}$ be an increasing function. Then f is differentiable almost everywhere. Moreover:

$$\int_a^b f'(x)dx \leq f(b) - f(a)$$

Definition 2.22. Given $f : [a, b] \to \mathbb{R}$, the *total variation* of f over $[a, b]$ is:

$$Vf := \sup\{\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|\}$$

where the supremum is taken over all possible partitions of $[a, b]$. A function f is of *bounded variation* if Vf is finite.

Theorem 2.23. A function $f : [a, b] \to \mathbb{R}$ is of bounded variation if and only if f is the difference of two monotone functions on $[a, b]$.

Definition 2.24. A function $f : [a, b] \to \mathbb{R}$ is *absolutely continuous* if $\forall \varepsilon > 0 \, \exists \delta > 0$ such that:

$$\sum_{i=1}^{n} |f(b_i) - f(a_i)| < \varepsilon$$

for all n and every disjoint collection of intervals $(a_1, b_1), \ldots, (a_n, b_n)$ with $\sum_{i=1}^{n} b_i - a_i < \delta$.

Theorem 2.25. Let $f : [a, b] \to \mathbb{R}$ be continuous and increasing. The following are equivalent:

- f is absolutely continuous on $[a, b]$
- f maps sets of measure 0 to sets of measure 0
- f is differentiable almost everywhere on $[a, b], f' \in L^1$ and:

$$\int_a^x f'(t)dt = f(x) - f(a)$$

3 Complex Analysis

3.1 $\mathbb{C}$ Review

Definition 3.1. For $z = x + iy$, we have:

- $\mathbb{C} = \{z = x + iy : x, y \in \mathbb{R}\}$, where $i^2 = -1$
- $\operatorname{Re} z = x$, and $\operatorname{Im} z = y$.
- $\overline{z} = x - iy$, the complex conjugate of z.
- $|z| = \sqrt{x^2 + y^2}$, the norm of z.

Definition 3.2. We say that $(z_n)_{n=1}^{\infty} \subset \mathbb{C}$ *converges* to z if and only if $\lim_{n\to\infty} |z_n - z| \to 0$, i.e., $\forall \varepsilon > 0 \, \exists N > 0$ such that $|z_n - z| < \varepsilon \, \forall n > N$.

Definition 3.3. We say that $\Omega \subset \mathbb{C}$ is *open* if and only if $\forall x \in \Omega \, \exists r > 0$ such that $\mathcal{B}_r(x) = \{z \in \mathbb{C} : |z - x| < r\} \subset \Omega$. We say that Ω is *closed* if and only if Ω^c is open.

Definition 3.4. A set $K \subset \mathbb{C}$ is *sequentially compact* if and only if every sequence $(x_j)_{j\in\mathbb{N}} \subset K$ has a convergent subsequence $(x_{j(t)})_{l\in\mathbb{N}}$ whose limit is in K.

Definition 3.5. Given $f : \Omega \subset \mathbb{C} \to \mathbb{C}$ we say that it is *continuous* at $z_0 \in \Omega$ if and only if $\forall \varepsilon > 0 \, \exists \, \delta$ such that $|z - z_0| < \delta$, with $z \in \Omega$ implies that $|f(z) - f(z_0)| < \varepsilon$.

Definition 3.6. Let $\Omega \subset \mathbb{C}$ be an open set and $z_0 \in \Omega$. We say that f is *complex differentiable* at z if and only if the limit:

$$\lim_{h\to 0} \frac{f(z+h) - f(z)}{h}$$

exists. We denote this limit by $f'(z)$.

Definition 3.7. We say that $f : \Omega \to \mathbb{C}$ is *analytic/holomorphic* in a neighbourhood U of z if it is complex differentiable everywhere in U. We say that f is *entire* if it is analytic in the whole of $\mathbb{C}$.

Theorem 3.8. Let $f : \Omega \subset \mathbb{C} \to \mathbb{C}$ with Ω open. f is complex differentiable of $z = a + ib \in \Omega$ if and only if f, when considered as a map from $\Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$ ($f(z) = u(z) + iv(z)$, where u and v are real valued functions) has a differential at the point (a, b) that satisfies the Cauchy-Riemann equations:

$$u_x = v_y \qquad u_y = -v_x$$

Theorem 3.9. Let $f, g : \Omega \subset \mathbb{C} \to \mathbb{C}$ be complex differentiable functions. Then (assuming $g \neq 0$ in the third expression, we have the familiar expressions:

$$(f+g)' = f' + g' \qquad (fg)' = f'g + fg' \qquad \left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2} \qquad (f(g))' = f'(g)g'$$

3.2 Power Series

Definition 3.10. The series $\sum_{n=0}^{\infty} a_n \in \mathbb{C}$ is *convergent* if and only if the sequence $S_N = \sum_{n=0}^{\infty}$ is convergent in $\mathbb{C}$.

Definition 3.11. The series $\sum_{n=0}^{\infty} a_n$, with $a_n \in \mathbb{C}$ is *absolutely convergent* if and only if the series $\sum_{n=0}^{\infty} |a_n|$ is convergent.

Theorem 3.12. Ratio Test: Consider $\sum_{n=0}^{\infty} a_n$. Then:

- If $\limsup \frac{|a_{n+1}|}{|a_n|} < 1$, then $\sum_{n=0}^{\infty} a_n$ is convergent.
- If $\limsup \frac{|a_{n+1}|}{|a_n|} > 1$, then $\sum_{n=0}^{\infty} a_n$ is divergent.

Theorem 3.13. Root Test: Consider $\sum_{n=0}^{\infty} a_n$. Then:

- If $\limsup |a_n|^{1/n} < 1$, then $\sum_{n=0}^{\infty} a_n$ converges.
- If $\limsup |a_n|^{1/n} > 1$, then $\sum_{n=0}^{\infty} a_n$ diverges.

Theorem 3.14. Given $(a_n)_{n=0}^{\infty}, \exists \, R \in [0, \infty]$ such that $\sum_{n=0}^{\infty} a_n z^n$ converges $\forall \, |z| < R$ and diverges $\forall \, |z| > R$. Furthermore:

$$R = \frac{1}{\limsup |a_n|^{1/n}}$$

Theorem 3.15. Let $a_n \neq 0 \, \forall \, n \geq N$, and assume that $\lim \frac{|a_{n+1}|}{|a_n|}$ exists. Then $\sum_{n=0}^{\infty} a_n$ has radius of convergence $R = \lim \frac{|a_n|}{|a_{n+1}|}$.

Theorem 3.16. Let $\sum_{n=0}^{\infty} a_n z^n$ have radius of convergence R. Then for $|z| < R$ the function $f(z) := \sum_{n=0}^{\infty} a_n z^n$ is differentiable and:

$$f'(z) = \sum_{n=0}^{\infty} n a_n z^{n-1}$$

Corollary 3.17. Let $\sum_{n=0}^{\infty} a_n z^n$ be a power series with radius of convergence $R > 0$. Then $f(z) := \sum_{n=0}^{\infty} a_n z^n$ is infinitely differentiable and moreover:

$$f^{(n)}(0) = a_n n! \qquad n = 0, 1, 2, \dots$$

Theorem 3.18. Let $\sum_{n=0}^{\infty} a_n z^n$ be a power series with radius of convergence $R > 0$. Then for every $r < R$ the sequence of functions:

$$f_k := \sum_{n=0}^{k} a_n z^n$$

converges uniformly in $|z| \leq r$.

Definition 3.19. We define the following power series for $z \in \mathbb{C}$:

- $e^z := \sum_{n=0}^{\infty} \frac{1}{n!} z^n$
- $\cos(z) := \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n}$
- $\cosh(z) := \sum_{n=0}^{\infty} \frac{1}{(2n)!} z^{2n}$
- $\sin(z) := \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}$
- $\sinh(z) := \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} z^{2n+1}$

Proposition 3.20. The following identities hold $\forall\, z \in \mathbb{C}$:

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2} \qquad \cosh(z) = \frac{e^{z} + e^{-z}}{2} \qquad \sin(z) = \frac{e^{iz} - e^{-iz}}{2} \qquad \sinh(z) = \frac{e^{z} - e^{-z}}{2}$$

Note: The following relationships are evident:

$$\cos(iz) = \cosh(z) \qquad \cosh(iz) = \cos(z) \qquad \sin(iz) = i\sinh(z) \qquad \sinh(iz) = i\sin(z)$$

Theorem 3.21. The exponential function e^z satisfies the following properties:

- $e^{z+w} = e^z e^w \,\forall\, z, w \in \mathbb{C}$.
- $e^z \neq 0 \,\forall z \in \mathbb{C}$.
- $e^z = 1$ if and only if $z = 2k\pi i$ for $k \in \mathbb{Z}$, and as a result, $e^{z+w} = e^z$ if and only if $w = 2k\pi i, k \in \mathbb{Z}$.

Definition 3.22. We define, for $z \neq 0$:

$$\arg(z) = \{\theta \in \mathbb{R} : z = |z|e^{i\theta}\}$$

Proposition 3.23. Properties of $\arg(z)$:

- $\arg(\alpha z) = \arg(z) \,\forall\, \alpha > 0$,
- $\arg(\alpha z) = \arg(z) + \pi = \{\theta + \pi : \theta \in \arg(z)\} \,\forall\, \alpha > 0$,
- $\arg(\overline{z}) = -\arg(z) = \{-\theta : \theta \in \arg(z)\}$,
- $\arg(\frac{1}{z}) = -\arg(z)$,
- $\arg(zw) = \arg(z) + \arg(w) = \{\theta + \phi : \theta \in \arg(z), \phi \in \arg(w)\}$.

3.3 Complex Integration and Contour Integrals

Definition 3.24. For a function $f : [a, b] \to \mathbb{C}$, we define:

$$\int_a^b f(t)dt = \int_a^b \operatorname{Re} f(t)dt + i \int_a^b \operatorname{Im} f(t)dt$$

Proposition 3.25. For every $f, g : [a, b] \to \mathbb{C}$ and every $\alpha, \beta \in \mathbb{C}$, we have:

- $\int_a^b [\alpha f + \beta g]dt = \alpha \int_a^b f(t)dt + \beta \int_a^b g(t)dt$,
- $\overline{\int_a^b f(t)dt} = \int_a^b \overline{f(t)}dt$,
- $|\int_a^b f(t)dt| \le \int_a^b |f(t)|dt$.

Definition 3.26. Given a function $f : \Omega \subset \mathbb{C} \to \mathbb{C}$ along the path $\Gamma \subset \Omega \subset \mathbb{C}$ parameterised by $\gamma : [a, b] \to \mathbb{C}$ is given by:

$$\int_\Gamma f dz = \int_a^b f(\gamma(t))\gamma'(t)dt = \int_a^b \operatorname{Re}(f(\gamma(t))\gamma'(t))dt + i \int_a^b \operatorname{Im}(f(\gamma(t))\gamma'(t))dt$$

Definition 3.27. If Γ is a union of n curves Γ_i, then we can define:

$$\int_\Gamma f dz := \sum_{j=1}^{n} \int_{\Gamma_j} f dz$$

Lemma 3.28. Let $\Gamma \subset \mathbb{C}$ be a curve, parameterised by $\gamma : [a, b] \to \mathbb{C}$, so $\gamma([a, b]) = \Gamma$. Given $f : \Omega \subset \mathbb{C} \to \mathbb{C}$:

- If γ^- represents the parameterisation of y in the opposite direction, then:

$$\int_{\gamma^-} f = -\int_\gamma f$$

  If a curve Γ has attached a sense of direction, we will call it a directed curve. In this case we will use $-\Gamma$ to denote the same curve swept in the opposite direction. Thus, reformulating the above:

$$\int_\Gamma dfz = -\int_{-\Gamma} f dz$$

- If $\tilde{\gamma} : [\tilde{a}, \tilde{b}] \to \mathbb{C}$ is another parameterisation of Γ that preserves the orientation, then:

$$\int_{\tilde{\gamma}} f = \int_\gamma f$$

  We refer to this fact as reparameterisation invariance.

Definition 3.29. Given $f : \mathbb{C} \to \mathbb{C}$ and a curve $\gamma : [a, b] \to \mathbb{C}$, we define:

$$\int_\gamma f d\overline{z} := \int_a^b f(\gamma(t))\overline{\gamma'(t)}dt$$

Note: In general:

$$\overline{\int_\gamma f(z)dz} \neq \int_\gamma \overline{f(z)}dz$$

Instead, we have:

$$\overline{\int_\gamma f(z)dz} = \int_\gamma \overline{f(z)}d\overline{z}$$

Theorem 3.30. Assume that $F : \Omega \subset \mathbb{C} \to \mathbb{C}$ is analytic and set $f(z) = \frac{dF}{dz}$. Let $\gamma : [a, b] \to \Omega$ be a curve. Then:

$$\int_\gamma f dz = F(\gamma(b)) - F(\gamma(a))$$

Proposition 3.31. For a function $f = u + iv$ and a curve $\gamma(t) = \gamma_1(t) + i\gamma_2(t)$, we have:

$$\int_\gamma f dz = \text{circulation}(\underline{f}) + i\text{flux}(\underline{f})$$

Where we have the vector field $\underline{f} = (u, -v)$ and the definitions for circulation and flux are taken from MA259 Multivariable Calculus.

Theorem 3.32. A set $\Omega \subset \mathbb{C}$ is *connected* if it cannot be expressed as a union of non-empty open sets Ω_1 and Ω_2 such that $\Omega_1 \cap \Omega_2 = \emptyset$. An open, connected set $\Omega \subset \mathbb{C}$ is called *simply connected* if every closed curve in Ω can be continuously deformed to a point. To put it simply, a simply connected domain has 'no holes'.

Theorem 3.33. Cauchy's Theorem: Let $f : \Omega \to \mathbb{C}$ be an analytic function, with Ω an open, simply connected domain. Let γ be a C^1 closed curve in Ω. Then:

$$\int_\gamma f(z) dz = 0$$

Theorem 3.34. Let $\Omega \subset \mathbb{C}$ be a region bounded by two simple curves γ_1 (the exterior curve) and γ_2 (the interior). Assume they are oriented positively, and let f be an analytic function in $\Omega \cup \gamma_1 \cup \gamma_2$. Then:

$$\int_{\gamma_1} f dz + \int_{\gamma_2} f dz = 0$$

Similarly, if we denote by γ_2^- the anti-clockwise parameterisation, then the result can be rephrased as:

$$\int_{\gamma_1} f dz = \int_{\gamma_2^-} f dz$$

Definition 3.35. Given a simple closed C^1 curve γ we denote by $I(\gamma)$ the *interior region* to γ. We denote by $O(\gamma)$ the *exterior region* to γ.

Theorem 3.36. Let $\gamma : [a, b] \to \mathbb{C}$ be a positively oriented simple closed C^1 curve. Assume that f is analytic in γ and on the interior of $\gamma, I(\gamma)$. Then:

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w - z} dw \qquad \forall\, z \in I(\gamma)$$

Theorem 3.37. Let $\gamma : [a, b] \to \mathbb{C}$ be a positively oriented simple closed C^1 curve. Assume that f is analytic in γ and on the interior of $\gamma, I(\gamma)$. Then $f^{(n)}(z)$ exists $\forall\, n \in \mathbb{N}$ and the derivative is given by:

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_\gamma \frac{f(w)}{(w - z)^{(n+1)}} dw \qquad \forall\, z \in I(\gamma)$$

Theorem 3.38. Taylor Series Expansion: Let f be an analytic function on $B_R(a)$ for $a \in \mathbb{C}, R > 0$. Then $\exists!$ constants $c_n, n \in \mathbb{N}$ such that:

$$f(z) = \sum_{n=0}^{\infty} c_n (z - a)^n \qquad \forall\, z \in B_R(a)$$

Moreover, the coefficients c_n are given by:

$$c_n = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{(w - a)^{n+1}} dw = \frac{f^{(n)}(a)}{n!}$$

where γ is any positively oriented simple closed curve that is contained in $B_R(a)$.

Theorem 3.39. Liouville's Theorem: Let $f : \mathbb{C} \to \mathbb{C}$ be an analytic, bounded function. Then f is constant.

Theorem 3.40. Fundamental Theorem of Algebra: Every non-constant polynomial p on $\mathbb{C}$ has a root, that is $\exists\, a \in \mathbb{C}$ such that $p(a) = 0$.

Theorem 3.41. Let $f_n : \Omega \to \mathbb{C}$ be a sequence of analytic functions on an open set Ω. If f_n converges uniformly to f, then f is analytic.

ANOTHER DISCLAIMER: Please note that this guide is a list of the key definitions and theorems you should be able to recall and utilise throughout the module. Proofs, calculations and examples, whilst omitted, are also very important for not just preparing for the exam but for preparing you for later modules in your degree. We highly recommend that you don't just use this guide as your sole revision resource. Please refer back to the printed lecture notes and your own notes for a full comprehension of the module, as this guide is more for helping factual recall, rather than understanding.

This guide would not be possible without our wonderful sponsors: