MA254

Theory of ODEs Revision Guide

Written by Benedetta Bertoli

Contents

Introduction

This revision guide for MA254 Theory of ODEs has been designed as an aid to revision, not a substitute for it. This guide is useful for revising through key definitions, theorems and some shorter proofs found in the course. However, a lot of the calculation methods and practical applications of the content of this module are omitted, for which it would be best to refer to the lectures and the online notes for said techniques.

Authors

Written by Benedetta Bertoli.
Based upon the lecture notes for MA254 Theory of ODEs, written by Dr. Claude Baesens at the University of Warwick.
Any corrections or improvements should be reported by email to comms@warwickmaths.org.

1 Existence and Uniqueness for IVPs

1.1 Picard's theorems

This module mainly focuses on examining the behaviour of solutions of initial value problems (IVP), i.e. problems of the form:

$$\begin{cases} \frac{dx}{dt}(t) = f(t, x(t)) \\ x(t_0) = x_0 \end{cases}$$

where $x : [t_0 - \tau, t_0 + \tau] \to \mathbb{R}^n$ is a differentiable function, $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function, $t_0 \in \mathbb{R}$, $x_0 \in \mathbb{R}^n$ and $\tau > 0$.

Lemma 1.1. $x : [t_0 - \tau, t_o + \tau] \to \mathbb{R}^n$ *satisfies the IVP above if and only if* x *is continuous and satisfies*

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds \qquad \text{for all } t \in [t_0 - \tau, t_0 + \tau].$$

Theorem 1.2 (Picard's theorem- global version)**.** Let $f : \mathbb{R} \times \mathbb{R}^n$ be continuous and assume $\exists L > 0$ such that

$$|f(s, u) - f(s, v)| \leq L|u - v| \text{ for all } s \in \mathbb{R} \text{ and } u, v \in \mathbb{R}^n$$

(so f is globally Lipschitz). Let $\tau < \frac{1}{L}$. Then, for all $x_0 \in \mathbb{R}^n$ there is a unique function $x : [-\tau, \tau] \to \mathbb{R}^n$ satisfying the IVP

$$\begin{cases} \frac{dx}{dt}(t) = f(t, x(t)) \\ x(t_0) = x_0 \end{cases}$$

Example 1.3. We can use this theorem to prove existence and uniqueness of solutions for a system of linear ODEs. Consider

$$\begin{cases} \frac{dx}{dt} = Ax \\ x(0) = x_0 \end{cases}$$

where A is an $n \times n$ matrix and $x_0 \in \mathbb{R}^n$. Then:

$$|Au - Av| = |A(u - v)| \leq \|A\||u - v|$$

for any $u, v \in \mathbb{R}^n$. So, for each choice of x_0, by theorem 1.2 the IVP above has a unique solution $t \mapsto x(t)$ defined for $t \in [-\tau, \tau]$, with $\tau \leq \frac{1}{\|A\|}$.

However, in many cases we have functions that are locally Lipschitz, but not globally Lipschitz. The following theorem describes the conditions under which we can ensure the existence and uniqueness of a solution in this case.

Theorem 1.4 (Local Picard's theorem)**.** Let $U \subset \mathbb{R} \times \mathbb{R}^n$ be open, with $(0, x_0) \in U$ and assume $f : U \to \mathbb{R}^n$ is continuous with $|f| \leq M$ and $|f(t, u) - f(t, v)| \leq L|u - v|$ for all $(t, u), (t, v) \in U$.

Then, there exists a $\tau \in \left(0, \frac{1}{L}\right)$ and a unique $x : [-\tau, \tau] \to \mathbb{R}^n$ satisfying $\frac{dx}{dt} = f(x, t)$ and $x(0) = x_0$.

Remark 1.5. This theorem also applies for any other inital time $t_0 \in R$.

Picard's theorem has some important implications:

- If $x_1 : I_1 \to \mathbb{R}^n$ and $x_2 : I_2 \to \mathbb{R}^n$ are both solutions to the same IVP, then $x_1(t) \equiv x_2(t)$ for all $t \in I_1 \cap I_2$.
- If the differential equation is autonomous (i.e. the function f does not depend on t), then solutions cannot cross.

1.2 Maximal interval of existence

Definition 1.6. The *maximal interval of existence* $J(t_0, x_0)$ is the largest interval of t that includes t_0 for which the solution $x(t)$ to the IVP $\dot{x} = f(t,x), x(t_0) = x_0$ exists.

Theorem 1.7. Let $U \in \mathbb{R} \times \mathbb{R}^n$ be open and $f : U \to \mathbb{R}^n$ be locally Lipschitz. Then there is a maximal open interval $J = (\alpha, \beta)$ with $t_0 \in J$ such that the IVP

$$\begin{cases} \dot{x} = f(t,x) \\ x(t_0) = x_0 \end{cases}$$

has a unique solution $x : J \to \mathbb{R}^n$

Theorem 1.8 (Unboundedness of solutions)**.** Let $U \subset \mathbb{R} \times \mathbb{R}^n$ be open (say $U = \mathbb{R} \times \tilde{U}$), and let $f : U \to \mathbb{R}^n$ be locally Lipschitz. Let $J = (\alpha, \beta)$ be the maximal interval of existence of the IVP $\dot{x} = f(t,x), x(t_0) = x_0$.

If β is finite, then for every compact set $K \subset \tilde{U}$, there is a $t \in [t_0, \beta]$ such that $x(t) \notin K$. Similarly, if α is finite then for all compact sets $K \subset \tilde{U}$, there is a $t \in (\alpha, t_0]$ such that $x(t) \notin K$.

From this theorem it follows that:

Corollary 1.9. *If β is finite, then either $\lim_{t\to\beta} x(t)$ does not exist, or $\lim_{t\to\beta} x(t)$ lies in the boundary of $\tilde{U}$.*

Remark 1.10. If $\lim_{t\to\beta} x(t)$ does not exist, we say $x(t)$ *blows up in finite time.*

When our differential equation is linear, however, we can be sure that finite time blow up does not happen.

Theorem 1.11. The IVP

$$\begin{cases} \dot{x} = A(t)x + b(t) \\ x(0) = x_0 \end{cases}$$

with $A : \mathbb{R} \to \mathbb{R}^{n\times n}$ continuous and $b : \mathbb{R} \to \mathbb{R}^n$ continuous, has a unique solution defined on $J = \mathbb{R}$.

When the maximum interval of existence of an IVP is $\mathbb{R}$, we say that such IVP has a *global solution.*

2 Linear systems with constant coefficients

In this section we focus on systems of linear differential equations, which can be written as:

$$\begin{cases} \dot{x} = Ax \\ x(0) = x_0 \end{cases}$$

where A is a constant $n \times n$ matrix and $x : \mathbb{R} \to \mathbb{R}^n$.

2.1 Matrix exponentials

We will soon see that such IVP has solution $e^{tA}x_0$. But what does it mean to take the exponential of a matrix?

Definition 2.1. For an $n \times n$ matrix, we define $e^{tA} = \sum_{k=0}^{\infty} \frac{(tA)^k}{k!}$

Remark 2.2. The series $\sum_{k=0}^{\infty} \frac{(tA)^k}{k!}$ converges absolutely for any A, so e^{tA} is well defined.

Before proving that $e^{tA}x_0$ is indeed the unique solution to the IVP, we need some technical lemmas telling us how to work with matrix exponentials.

Lemma 2.3. *Let A, B, T be $n \times n$ matrices and T be invertible. Then*

- *If $B = T^{-1}AT$ then* $\exp(B) = T^{-1}\exp(A)T$

- *If* $AB = BA$ *then* $\exp(A+B) = \exp(A)\exp(B)$
- $\exp(-A) = (expA)^{-1}$

Theorem 2.4. The solution to the IVP

$$\begin{cases} \dot{x} = Ax \\ x(0) = x_0 \end{cases}$$

is $e^{tA}x_0$.

Proof. It is possible to prove this using the definition of derivative to show that $\frac{d}{dt}e^{tA} = Ae^{tA} = e^{tA}A$. Alternatively, using the power series definition of e^{tA}:

$$\begin{aligned} \frac{d}{dt}\sum_{n=0}^{\infty}\frac{t^nA^n}{n!}x_0 &= \sum_{n=1}^{\infty}\frac{nt^{n-1}}{n!}A^nx_0 \\ &= \sum_{n=0}^{\infty}\frac{t^nA^{n+1}}{n!}x_0 \\ &= Ae^{tA}x_0 \end{aligned}$$

Moreover, it is easy to see that at $t = 0$, $e^{tA}x_0 = e^0x_0 = x_0$. □

In general, it is not easy to compute e^{At} from the definition. We can instead compute it for a simpler related matrix and then use lemma 2.3. Recall from Algebra 1 that for every real matrix A, we can find an invertible matrix T such that $A_J := T^{-1}AT$ is in Jordan normal form. For a matrix A_J in Jordan normal form, it is possible to calculate $(tA_J)^k$ inductively, and consequently to calculate e^{tA_J}. Then, applying lemma 2.3, we have that

$$\exp(tA) = T\exp(t(T^{-1}AT))T^{-1} = T\exp(tA_J)T^{-1}.$$

2.2 Classification of 2-dimensional linear systems

By studying the eigenvalues of the matrix A in the IVP

$$\begin{cases} \dot{x} = Ax \\ x(0) = x_0 \end{cases}$$

it is possible to understand the behaviour of the solutions by drawing a *phase portrait* and analysing the stability of the origin. The eigenvalues of a matrix A are given by solving the characteristic equation $\lambda^2 - \text{tr}(A)\lambda + \det(A) = 0$ where $\text{tr}(A)$ denotes the trace of A (which is the sum of its diagonal entries), and $\det(A)$ its determinant.

Then:

$$\lambda_{\pm} = \frac{\text{tr}(A) \pm \sqrt{(\text{tr}(A))^2 - 4\det(A)}}{2}$$

are the eigenvalues. Let $\Delta := \text{tr}(A)^2 - 4\det(A)$. If:

- $\Delta > 0$: we have two real eigenvalues λ, μ. If $\lambda, \mu > 0$, the origin is an *unstable node*. If $\lambda, \mu < 0$, the origin is a *stable node*. Otherwise, if the eigenvalues have opposite signs, we say that the origin is a *saddle*.

- $\Delta < 0$: we have two complex conjugate eigenvalues λ, μ. Then the origin is a *focus* and we say it is unstable if the real part of the eigenvalues is positive, and stable if it is negative. If the real part is 0, we say the origin is a *centre*.

- $\Delta = 0$: we have one repeated real eigenvalue λ. In this case, we have an *improper node*, which is unstable if $\lambda > 0$ and stable if $\lambda < 0$.

3 Qualitative theory of ODEs

From now on, we restrict ourselves to autonomous differential equations $\dot{x} = f(x)$, with f continuously differentiable (so solutions to the IVP with $x(0) = x_0$ exist and are unique). The state space X will be $\mathbb{R}^n$ most of the time.

3.1 General concepts

We would like to study the behaviour of all solutions of the IVP together, rather than having to analyse each initial value separately. To this end, we define the *flow* of $\dot{x} = f(x)$ to be the map $\phi : X \times \mathbb{R} \to X : (x_0, t) \mapsto \phi_t(x_0)$, where $\phi_t(x_0)$ is the solution to the initial value problem at time t with initial value x_0.

Remark 3.1. This function satisfies $\phi_s(\phi_t(x_0)) = \phi_{s+t}(x_0) = \phi_{t+s}(x_0)$.

In order to analyse the qualitative behaviour of ODEs, we need to first set out some definitions.

Definition 3.2. The solutions $t \mapsto \phi_t(x)$ are called *orbits* or *trajectories*.

Definition 3.3.
- The *orbit* of a point $x \in X$ is the set $O(x) := \{\phi_t(x) : t \in \mathbb{R}\}$
- The *forward orbit* of x is the set $O^+(x) := \{\phi_t(x) : t \geq 0\}$
- The *backward orbit* of x is the set $O^-(x) := \{\phi_t(x) : t \leq 0\}$

Definition 3.4. A point $x_0 \in X$ is a *fixed point* of $\dot{x} = f(x)$ if $f(x_0) = 0$

Definition 3.5. A *periodic point* x is a point such that $\phi_T(x) = x$ for some $T > 0$ and $\phi_t(x) \neq x$ for $0 < t < T$. Equivalently, x is a periodic point if for all $t, \phi_{t+T}(x) = \phi_t(x)$. Then, $O(x)$ is called a *periodic orbit.*

Definition 3.6. Let x be a fixed point. If $\phi_t(y) \to x$ as $t \to \pm\infty$, then $O(y) = \{\phi_t(y) : -\infty < t < \infty\}$ is called a *homoclinic orbit.*

Definition 3.7. Let $x_0 \neq x_1$ be fixed points. If

$$\phi_t(y) \to \begin{cases} x_0 \text{ as } t \to \infty \\ x_1 \text{ as } t \to -\infty \end{cases}$$

then $O(y)$ is called a *heteroclinic orbit.*

Definition 3.8.
- A subset $\Lambda \subset X$ is said to be *invariant* under a flow ϕ if $x \in \Lambda$ implies $\phi_t(x) \in \Lambda$ for all $t \in \mathbb{R}$.
- $\Lambda \subset X$ is said to be *forward invariant* under a flow ϕ if $x \in \Lambda$ implies $\phi_t(x) \in \Lambda$ for all $t \geq 0$.
- $\Lambda \subset X$ is said to be *backward invariant* under a flow ϕ if $x \in \Lambda$ implies $\phi_t(x) \in \Lambda$ for all $t \leq 0$.

Definition 3.9. A point $y \in X$ is said to be an *ω-limit point* of $x \in X$ for a flow ϕ if $\phi_{t_k} \to y$ as $k \to \infty$ for some increasing sequence of time t_k. The set $\omega(x) := \{y \in X : \exists t_k \to \infty \text{ such that } \phi_{t_k}(x) \to y \text{ as } t \to \infty\}$ is called the *ω-limit set of x.*

Similarly, the $\alpha - $ *limit set* of x is the set of all limit points of the backward orbit of x : $\alpha(x) := \{y \in X : \exists (t_k) \to -\infty \text{ such that } \phi_{t_k}(x) \to y \text{ as } k \to \infty\}$

The omega limit set has some interesting properties.

Proposition 3.10. *$\omega(x)$ is closed and bounded.*

Proposition 3.11. *If the forward orbit of a point $x \in \mathbb{R}^n$ lies in a compact subset K of $\mathbb{R}^n$, then $\omega(x)$ is non-empty and $\omega(x) \subset K$.*

We now move on to the concept of stability of a set.

Definition 3.12. An invariant set Λ is said to be *Lyapunov stable* if given any neighbourhood U of Λ there is a neighbourhood V of Λ such that $x \in V \Rightarrow \phi_t(x) \in U$ for all $t \geq 0$.

Informally speaking, this means that "if we start near, we stay near".

Definition 3.13. An invariant set Λ is said to be *asymptotically stable* (or *attracting*, or *attractor*), if:

- Λ is Lyapunov stable, and
- there is a neighbourhood V of Λ such that for all $x \in V, \phi_t(x) \to \Lambda$ as $t \to \infty$, meaning

$$d(\phi_t(x), \Lambda) := \inf\{d(\phi_t(x), y) : y \in \Lambda\} \to 0 \text{ as } t \to \infty.$$

Informally speaking, "start near, stay near and get closer".

Definition 3.14. An invariant set that is attracting under time reversal is called a *repellor* or a *repelling set*.

However, we may have sets satisfying the second point of definition 3.13 but which are not Lyapunov stable. For these, sets, we introduce a new definition.

Definition 3.15. An invariant set is said to be *eventually attracting* if there exists a neighbourhood V of Λ such that for all $x \in \Lambda, \phi_t(x) \to \Lambda$ as $t \to \infty$.

Definition 3.16. If Λ is an attracting invariant set, then its *basin of attraction* is defined as

$$B(\Lambda) := \{x \in X : d(\phi_t(x), \Lambda) \to 0 \text{ as } t \to \infty\}$$

If $B(\Lambda) = X$, then we say Λ is a *global attractor*.

In other words, the basin of attraction of an invariant set Λ is the set of all points whose orbits tend to Λ. $B(\Lambda)$ is an open set.

Example 3.17. Consider a linear system $\dot{x} = Ax$, $X = \mathbb{R}^n$, where A is a constant $n \times n$ matrix. If all the eigenvalues of A have negative real parts, then the origin is a global attractor.

3.2 One-dimensional dynamics

This short section analyses two interesting facts about one dimensional ODEs. Let $\dot{x} = f(x)$ on $X = \mathbb{R}$, with f a Lipschitz continuous function. Let ϕ be its flow, which is well defined since f is Lipschitz.

Proposition 3.18. *The orbits of the equation above consists of fixed points, heteroclinic orbits joining fixed points, and orbits which tend to $\pm\infty$ or come from $\pm\infty$.*

Proposition 3.19. *Suppose $f \neq 0$ on $I = (a, b)$ and $f(a) = f(b) = 0$. Then, for all $x \in I$, $\phi_t(x)$ is defined for all $t \in \mathbb{R}$ and is monotonic. In particular:*

- *if $f > 0$ on I, $\phi_t(x)$ is increasing and it tends to b as $t \to \infty$, and to a as $t \to -\infty$;*
- *if $f < 0$ on I, $\phi_t(x)$ is decreasing and it tends to a as $t \to \infty$, and to b as $t \to -\infty$.*

3.3 Conservative systems

Definition 3.20. A system $\dot{x} = f(x)$ is said to be *conservative* if there exists a non-trivial C^1 function $H : X \to \mathbb{R}$ that is constant along all orbits, i.e. for all $x \in X$, $\frac{d}{dt}H(\phi_t(x)) = 0$ at $t = 0$.

By "non-trivial", we mean a function H whose derivative is non-zero almost everywhere.

Example 3.21. An important example of a conservative system is the ideal pendulum, represented by the ODE $\ddot{\theta} + \sin\theta = 0$, which can be written as a system of two first order equations as follows:

$$\begin{cases} \dot{\theta} = p \\ \dot{p} = -\sin\theta \end{cases}$$

This system is conservative with $H(x) = \frac{p^2}{2} - \cos\theta$, which represents the energy of the pendulum. Indeed, we have:

$$\frac{d}{dt}H(x(t)) = \frac{\partial H}{\partial p}\dot{p} + \frac{\partial H}{\partial \theta}\dot{\theta} = p(-\sin\theta) + (\sin\theta)p = 0.$$

Remark 3.22. Finding a function H whose derivative is zero along orbits can be very useful when drawing phase portraits. In fact, since H is constant along orbits, these must lie inside level sets of H, so analysing the level sets of H suffices to understand the behaviour of the solutions of the initial value problem.

Definition 3.23. A system of the form

$$\begin{cases} \dot{x} = \frac{\partial H}{\partial y}(x,y) \\ \dot{y} = -\frac{\partial H}{\partial x}(x,y) \end{cases}$$

where H is a C^1 function, is called a *Hamiltonian system*.

Hamiltonian systems are conservative since

$$\frac{dH}{dt} = \frac{\partial H}{\partial x}\dot{x} + \frac{\partial H}{\partial y}\dot{y} = \frac{\partial H}{\partial x}\frac{\partial H}{\partial y} - \frac{\partial H}{\partial y}\frac{\partial H}{\partial x} = 0.$$

3.4 Lyapunov functions

If there is a function $H : X \to \mathbb{R}$ which is non-increasing along orbits, this may also be useful for describing system dynamics.

Definition 3.24. A *Lyapunov function* for a flow $\phi : X \times \mathbb{R} \to X$ is a continuous function $H : X \to \mathbb{R}$ such that for all $x \in X$ and $t \geq 0, H(\phi_t(x)) \leq H(x)$, i.e. H is non-increasing along orbits.

If H is continuously differentiable, it suffices to check that $\frac{dH}{dt} \leq 0$ along orbits.
We now come to a very useful theorem about limit sets.

Theorem 3.25 (La Salle's invariance principle)**.** If H is a Lyapunov function for a flow ϕ, then for all $x \in X$ there exists a $c \in \mathbb{R}$ such that $\omega(x) \subset M_c := \{y \in X : H(\phi_t(y)) = c \text{ for all } t \geq 0\}$.

Remark 3.26. The set $M_c := \{y \in X : H(\phi_t(y)) = c \text{ for all } t \geq 0\}$ is the set of all forward orbits along which H is constant (and equal to c).

This is very useful to find $\omega-$ limit sets, especially if you have a Lyapunov function H that is constant only at the fixed points. In this case, La Salle's invariance principle would imply that the orbit of any point in the system tends to one of the fixed points (or to a set of fixed points). We now look at how to apply this principle to gradient systems.

Definition 3.27. A *gradient system* is a system of the form $\dot{x} = -\nabla V(x)$, where $V : X \to \mathbb{R}$ is a C^2 function.

For such systems, V is a Lyapunov function; you can check that $\frac{dV}{dt} = -|\nabla V(x)|^2$. Hence, $\frac{dV}{dt} < 0$, except at points where $\nabla V = 0$, which are the fixed points. Therefore, La Salle's invariance principle implies that for any $x \in X$, $\omega(x)$ is a subset of a set of fixed points with the same value of V.

3.5 Dynamics near the fixed points

Sometimes our ODEs can get complicated, making it hard (or even impossible) to find their solutions. However, in some cases, looking at only the linear terms of the ODEs can be useful to understand the behaviour of the solutions near a fixed point. This section analyses how and when we can do this.

Consider the ODE $\dot{x} = f(x)$ on some state space X, where f is continuously differentiable. Let $\bar{x}$ be a fixed point of the equation. By Taylor's theorem

$$f(x) = f(\bar{x}) + Df(\bar{x})(x - \bar{x}) + O(|x - \bar{x}|)$$

where $O(|x - \bar{x}|)$ denotes the non linear (i.e. higher order) terms of the expansion, and $f(\bar{x}) = 0$, since $\bar{x}$ is a fixed point. We introduce a change of coordinates $x = \bar{x} + y$, which corresponds to putting $\bar{x}$ at the origin. Then:

$$\dot{\bar{x}} + \dot{y} = \dot{x} = f(x) = Df(\bar{x})y + O(|y|).$$

Since $\frac{O(|y|)}{|y|} \to 0$ as $y \to 0$, we can focus our attention on the equation:

$$\dot{y} = Df(\bar{x})y$$

which is called the *linearised dynamics* of $\dot{x} = f(x)$ about the fixed point $\bar{x}$.

Definition 3.28. We say a fixed point $\bar{x}$ is *hyperbolic* if $\mathrm{Re}(\lambda) \neq 0$ for each eigenvalue λ of $Df(\bar{x})$.

We will see that if $\bar{x}$ is a hyperbolic fixed point, then the local phase portrait near $\bar{x}$ can be well approximated by the one of the linearised system.

There are three types of hyperbolic fixed points, depending on the eigenvalues λ of $Df(\bar{x})$:

- If all eigenvalules of $Df(\bar{x})$ have $\mathrm{Re}(\lambda) < 0$, we say $\bar{x}$ is a *sink*.
- If all eigenvalues of $Df(\bar{x})$ have $\mathrm{Re}(\lambda) > 0$, we say $\bar{x}$ is a *source*.
- Otherwise, we say $\bar{x}$ is a *saddle point*.

An important property of these fixed points is that sinks are attracting and sources are repelling. More formally:

Theorem 3.29. Let $\bar{x}$ be a sink of $\dot{x} = f(x)$ on $X = \mathbb{R}^n$. Then $\bar{x}$ is attracting. More precisely, suppose $\mathrm{Re}(\lambda) < -a$ for some $a > 0$, for all eigenvalues λ of $Df(\bar{x})$. Then, there exists a neighbourhood V of $\bar{x}$ such that:

- $\phi_t(x)$ is defined and in V for all $x \in V$ and $t \geq 0$;
- There is a Euclidean norm $\|\cdot\|_B$ on $\mathbb{R}^n$ such that

  $$\|\phi_t(x) - \bar{x}\|_B \leq e^{-at}\|x - \bar{x}\|_B$$

  for all $x \in V$, $t \geq 0$;
- There is a constant $C > 0$ such that

  $$|\phi_t(x) - \bar{x}| \leq Ce^{-at}|x - \bar{x}|$$

  for all $x \in V$ and $t \geq 0$

where $|\cdot|$ denotes the usual norm on $\mathbb{R}^n$. In particular, $\phi_t(x) \to \bar{x}$ as $t \to \infty$ for all $x \in V$.

By reversing time, if all eigenvalues have real part greater than 0, then $\bar{x}$ is repelling (defined as attracting in negative time).

In some circumstances, we can prove that some fixed points that are not sinks are attracting.

Theorem 3.30. (Lyapunov's first and second stability theorems) Let ϕ be a continuous flow on X with fixed point $\bar{x}$. Let N be a neighbourhood of $\bar{x}$ and $H : X \to \mathbb{R}$ be a continuous function which is non-increasing along orbits in N. Then

- (*Lyapunov's first stability theorem*) $\bar{x}$ is Lyapunov stable.
- (*Lyapunov's second stability theorem*) if further H is decreasing along orbits in $N \setminus \{\bar{x}\}$ then $\bar{x}$ is asymptotically stable (attracting).

Example 3.31. Consider $\dot{x} = -x^3$ on state space $\mathbb{R}$. Use $H(x) = \frac{x^2}{2}$. Then

$$\frac{dH}{dt} = \frac{dH}{dx}\dot{x} = -x^4 < 0 \text{ for all } x \neq 0.$$

Hence, by Lyapunov's second stability theorem, the origin is attracting.

We also have a way of checking if a fixed point is unstable.

Theorem 3.32. Let $\bar{x}$ be a fixed point for $\dot{x} = f(x)$ on $E = \mathbb{R}^n$, where f is continuously differentiable. If $Df(\bar{x})$ has an eigenvalue λ with $\mathrm{Re}(\lambda) > 0$, then $\bar{x}$ is unstable.

We now move onto the stable and unstable manifold theorem. Before starting to analyse it, note that the definition of manifold is beyond the scope of this module. Since all applications of this theorem will be done in 1 or 2 dimensions, you only need to remember that one-dimensional manifolds are curves, and two dimensional manifolds are surfaces.

Let $\bar{x}$ be a hyperbolic fixed point of $\dot{x} = f(x)$ on $X = \mathbb{R}^n$, where f is continuously differentiable, and let

$$\dot{x} = Df(\bar{x})x \tag{1}$$

be its linearised dynamics on $E = \mathbb{R}^n$.

Denote by E^s the subspace of E spanned by the eigenvectors of $Df(\bar{x})$ associated to the eigenvalues λ which have $\mathrm{Re}(\lambda) < 0$. Let $d_s := dim(E^s)$.

Similarly, denote by E^u the subspace of E spanned by the eigenvectors with real parts greater than 0, and let $d_u := dim(E^u)$. Each of E^s and E^u is invariant under the linearised dynamics. In particular:

$$E^s = \{x \in E : O^+(x) \to 0 \text{ under } (1)\}$$

is called the *stable subspace* and

$$E^u = \{x \in E : O^-(x) \to 0 \text{ under } (1)\}$$

is called the *unstable subspace*.

Remark 3.33. The stable subspace is not stable in the Lyapunov sense.

Definition 3.34. The *stable manifold* $W^s_{\bar{x}}$ and the *unstable manifold* $W^u_{\bar{x}}$ of the fixed point $\bar{x}$ are defined to be the sets:

$$W^s_{\bar{x}} = \{x \in X : \phi_t(x) \to \bar{x} \text{ as } t \to \infty\}$$

$$W^u_{\bar{x}} = \{x \in X : \phi_t(x) \to \bar{x} \text{ as } t \to -\infty\}$$

In other words, the stable and unstable manifolds are the non linear equivalent of the stable and unstable subspaces.

We can now state the theorem.

Theorem 3.35 (Stable and unstable manifold theorem)**.** *Let $\bar{x}$ be a fixed point for $\dot{x} = f(x)$ on $E = \mathbb{R}^n$. Let $E^s = \{x \in E : O^+(x) \to 0\}$ and $E^u = \{x \in E : O^-(x) \to 0\}$ be the stable and unstable subspace respectively. Then, there exist a stable manifold $W^s_{\bar{x}}$ of dimension d_s and an unstable manifold $W^u_{\bar{x}}$ of dimension d_u, containing $\bar{x}$, that are tangent to $\bar{x} + E^s$ and $\bar{x} + E^u$, respectively, at $\bar{x}$.*

How can we apply the stable and unstable (SUM) theorem to two dimensional systems?

Suppose our fixed point $\bar{x}$ is a saddle, and the eigenvalues of $\dot{x} = Df(\bar{x})x$ are $\lambda, -\mu$, where both $\lambda, \mu > 0$. Then, the SUM theorem implies that there are precisely two orbits (+ itself) that converge to $\bar{x}$ as $t \to +\infty$, and they approach tangentially to the stable eigenvector E^s (corresponding to the eigenvalue $-\mu$), one from each side.

Similarly, it also implies that there are precisely two orbits (+itself) that converge to $\bar{x}$ as $t \to -\infty$, and they approach tangentially to the unstable eigenvector E^u (corresponding to the eigenvalue λ), one from each side.

We have seen that near a hyperbolic fixed point, the phase portrait of a system is similar to the one of its linearised dynamics. What can we say about the relationship between the orbits of such systems?

Theorem 3.36 (Hartman-Grobman's theorem)**.** *Suppose $\bar{x}$ is a hyperbolic fixed point of $\dot{x} = f(x)$ on $\mathbb{R}^n$, with f continously differentiable. Then, there exists a homeomorphism h from a neighbourhood U of $\bar{x}$ to a neighbourhood V of $0 \in \mathbb{R}^n$ such that $h \circ \phi_t = \psi_t \circ h$, where ϕ is the flow associated to $\dot{x} = f(x)$ and ψ is the flow associated to its linearisation at $\bar{x}$, which is $\dot{\xi} = Df(\bar{x})\xi$.*

In other words, it is possible to "match up" orbits of the linearised systems to those of the non-linear one.

We conclude this section with an important observation: the linearised system is very useful if the fixed point we are looking at is hyperbolic. However, if the fixed point is non-hyperbolic (i.e. $Df(\bar{x})$ has at least one eigenvalue λ with $\mathrm{Re}(\lambda) = 0$) the non-linear parts of the ODE may give rise to very different phase portaits, meaning linearised dynamics are not as useful.

3.6 Global phase portraits

In this section we examine the Lotka-Volterra model, used in population dynamics to analyse the interaction between species. Lotka-Volterra models are of the form:

$$\begin{cases} \dot{x} = x(A - a_1 x + b_1 y) \\ \dot{y} = y(B + b_2 x - a_2 y) \end{cases}$$

where $A, B, a_1, a_2 > 0$ are constants, and $x, y \geq 0$. Depending on the signs of b_1 and b_2, these equations can be used to model different types of interaction:

- $b_i > 0, i = 1, 2$ models a symbiotic relationship, where the interaction between the two species enhances the growth rate of each population, for example bees and flowers;
- $b_i < 0, i = 1, 2$ models competition between two species for the same resource, for example sheeps and rabbits competing for grass;
- $b_1 > 0, b_2 < 0$ models predator-prey interaction, where x is the predator and y the prey. An example is sheeps and wolves.

How do we draw the global phase portrait for this type of model?

- Step 1: find the fixed points and their linear stability. To do this, it suffices to set $\dot{x} = \dot{y} = 0$, and then calculate the eigenvalues of $Df(x, y)$ for each solution (x, y) (see section 3.5).
- Step 2: plot these fixed points on your phase portrait and draw the orbits nearby, whose behaviour you can deduce from the stability of the fixed points.
- Step 3: "join the dots" using the fixed points, nullclines (i.e. lines where $\dot{x} = 0$ or $\dot{y} = 0$) and the direction of the vector field.

3.7 Periodic orbits

Most of this section will be set in $\mathbb{R}^2$, since for some of the following theorems we require the existence of Jordan curves (i.e. curves that divide the state space into two disjoint pieces).

Theorem 3.37 (Poincaré-Bendixson's theorem)**.** Let $\dot{x} = f(x)$ on $X = \mathbb{R}^2$, with f continuously differentiable and associated flow ϕ. If $\phi_t(x) \in K \subset X$ for all $t \geq 0$ with K compact, then either:

- $\omega(x)$ contains a fixed point or
- $\omega(x)$ is a periodic orbit.

We can use this theorem to prove existence of periodic orbits. If there is a compact forward invariant region without any fixed points, then the Poincaré-Bendixson theorem implies that $\omega(x)$ is a periodic orbit for each x in the region.

Definition 3.38. A periodic orbit γ such that there is an $x \notin \gamma$ with $\omega(x) = \gamma$ or $\alpha(x) = \gamma$ is called a *limit cycle*.

There are also theorems to prove the non-existence of periodic orbits. First, recall the divergence theorem (covered in multivariable calculus).

Theorem 3.39. (Divergence theorem) Let Γ be a simple closed curve in $\mathbb{R}^2$ enclosing a region A; let $f : \mathbb{R}^2 \to \mathbb{R}^2$ and $g : \mathbb{R}^2 \to \mathbb{R}$ be continuously differentiable. Then

$$\int_\Gamma g\,(f \cdot n)\,dl = \iint_A \nabla \cdot (gf)\,dxdy$$

where n denotes the outward unit normal vector.

We can use this to prove:

Theorem 3.40 (Dulac's criterion)**.** If $\dot{x} = f(x)$ in $\mathbb{R}^2$ and there exists a continuous function $g : \mathbb{R}^2 \to \mathbb{R}$ such that $\nabla \cdot (gf)$ is continuous and non-zero on some simply connected domain D, then no periodic orbit can lie entirely in D.

Proof. Suppose there is a periodic orbit γ which lies entirely in D. Then, denoting by A the area bounded by γ:

$$\iint_A \nabla \cdot (gf)\, dxdy \neq 0$$

since $\nabla \cdot (gf)$ is continuous and non-zero in D. But f is tangent to γ since γ is a trajectory, so $f \cdot n = 0$ along γ. Therefore

$$\int_\gamma g\,(f \cdot n)\, dl = 0$$

which contradicts the divergence theorem. □

Remark 3.41. The function g in the theorem above is referred to as a *weighing function*. Moreover, if $g \equiv 1$, this theorem is referred to as the *divergence test*.

Example 3.42. We now apply this theorem to Lotka-Volterra models from the previous section.

$$\begin{cases} \dot{x} = x(A - a_1 x + b_1 y) \\ \dot{y} = y(B + b_2 x - a_2 y) \end{cases} \quad \text{where } A, B, a_1, a_2 > 0$$

We use the weighing function $g(x, y) = \frac{1}{xy}$. Then:

$$gf = \left(\frac{1}{y}(A - a_1 x + b_1 y), \frac{1}{x}(B + b_2 x - a_2 y) \right)$$

And

$$\nabla \cdot (gf) = -\frac{a_1}{y} - \frac{a_2}{x} < 0 \text{ for all } x, y > 0.$$

Therefore, by Dulac's criterium, there are no periodic orbits in the positive quadrant.

Remark 3.43. Taking $a_1, a_2 = 0$ in the system above gives a modified Lotka-Volterra model which has cyclic populations.

We now move our attention to Liénard systems. Consider the second order ODE

$$\ddot{x} + f(x)\dot{x} + g(x) = 0. \tag{2}$$

We already know how to write this as a system of two first order ODEs by setting $y = \dot{x}$. However, we can do this in another way by introducing the *Liénard variable* $y = \dot{x} + F(x)$, where

$$F(x) := \int_0^x f(s) ds.$$

Then, (2) can be rewritten as

$$\begin{cases} \dot{x} = y - F(x) \\ \dot{y} = -g(x) \end{cases}$$

which is referred to as a *Liénard system.*

Example 3.44. Consider the Van der Pol equation $\ddot{x} + \beta(x^2 - 1)\dot{x} + x = 0$, with $\beta > 0$. In the definition above, we take $F(x) = \beta \left(\frac{x^3}{3} - x \right)$ and $g(x) = x$. Then, the equation is transformed to

$$\begin{cases} \dot{x} = y - \beta \left(\frac{x^3}{3} - x \right) \\ \dot{y} = -x \end{cases}$$

Under some conditions, we can ensure existence and uniqueness of limit cycles (see 3.38) in Liénard's systems.

Theorem 3.45 (Liénard's theorem)**.** Let

$$\begin{cases} \dot{x} = y - F(x) \\ \dot{y} = -g(x) \end{cases}$$

be a Liénard's system.

Let F and g be C^1 functions and assume:

- F and g are odd functions (so $F(0) = g(0) = 0$);
- $g'(x) > 0$ for all $x \in \mathbb{R}$;
- a is the unique positive zero of F and $F(x) < 0$ for $0 < x < a$;
- F increases monotonically for $x > a$ and $F(x) \to \infty$ as $x \to \infty$.

Then, the Liénard system has a unique and stable limit cycle.